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$\mathrm{AdS}_4/\mathrm{CFT}_3$ duals from M2-branes at hypersurface singularities and their deformations

Dario Martelli[a] **and James Sparks**[b]

[a] *Swansea University,*
Singleton Park, Swansea, SA2 8PP, U.K.
[b] *Mathematical Institute, University of Oxford,*
24-29 St Giles', Oxford OX1 3LB, U.K.

E-mail: d.martelli@swansea.ac.uk, sparks@maths.ox.ac.uk

Abstract: We construct three-dimensional $\mathcal{N}=2$ Chern-Simons-quiver theories which are holographically dual to the M-theory Freund-Rubin solutions $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ (with or without torsion G-flux), where $V_{5,2}$ is a homogeneous Sasaki-Einstein seven-manifold. The global symmetry group of these theories is generically $\mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{U}(1)_R$, and they are hence non-toric. The field theories may be thought of as the $n=2$ member of a family of models, labelled by a positive integer n, arising on multiple M2-branes at certain hypersurface singularities. We describe how these models can be engineered via generalized Hanany-Witten brane constructions. The $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ solutions may be deformed to a warped geometry $\mathbb{R}^{1,2}\times T^*S^4/\mathbb{Z}_k$, with self-dual G-flux through the four-sphere. We show that this solution is dual to a supersymmetric mass deformation, which precisely modifies the classical moduli space of the field theory to the deformed geometry.

Keywords: Brane Dynamics in Gauge Theories, AdS-CFT Correspondence, Chern-Simons Theories, M-Theory

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Contents

JHEP12(2009)017

1 Introduction

The work of Bagger and Lambert [1] (see also [2]) has led to new insights into the low-energy physics of M2-branes. In [1] an explicit three-dimensional $\mathcal{N}=8$ supersymmetric gauge theory was constructed, a theory which was later shown to be a Chern-Simons-matter theory [3]. Following this work, Aharony, Bergman, Jafferis, and Maldacena (ABJM) [4] have constructed a class of three-dimensional Chern-Simons-quiver theories with generically $\mathcal{N}=6$ supersymmetry (enhanced to $\mathcal{N}=8$ for Chern-Simons levels $k=1,2$), and argued that these are holographically dual to the M-theory backgrounds $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, or their reduction to Type IIA string theory. This has renewed interest in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, opening the way for the construction of many new examples of this duality, in which Chern-Simons theories are believed to play a key role [5].

An interesting generalization of the ABJM duality is to consider theories with less supersymmetry. For example, the case of $\mathcal{N}=2$ (4 real supercharges) is analogous to minimal $\mathcal{N}=1$ supersymmetry in four dimensions. In the latter case, when the gauge theories are engineered by placing D3-branes at Calabi-Yau singularities the natural candidate holographic duals are given by Type IIB string theory on $\mathrm{AdS}_5 \times Y^5$, where Y^5 is a Sasaki-Einstein five-manifold. It can similarly be argued [6–8] that a large class of Chern-Simons-matter theories should be dual to $\mathcal{N}=2$ Freund-Rubin vacua of M-theory. This duality, for toric theories, has been studied in many papers — see, for example, [9].

In this paper we will discuss a three-dimensional Chern-Simons-quiver theory that we conjecture to be the holographic dual of M-theory on $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$, with N units of quantized G-flux, where $V_{5,2}$ (also known as a *Stiefel* manifold) is a homogeneous Sasaki-Einstein seven-manifold. This can be thought of as the near-horizon limit of N M2-branes placed at the Calabi-Yau four-fold singularity

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,, \qquad z_i \in \mathbb{C}\,, \tag{1.1}$$

which is clearly a generalization of the well-known conifold singulariy in six dimensions. Indeed, Klebanov and Witten mentioned this generalization in their seminal paper [10], concluding with the sentence: "*We hope it will be possible to construct a three-dimensional field theory corresponding to M2-branes on (1.1).*" In the present paper we will realize this hope. We propose[1] that the three-dimensional field theory in question is an $\mathcal{N}=2$ Chern-Simons-quiver theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, generalizing the ABJM model. The matter content and superpotential will be presented shortly in section 2; see figure 1 and equation (2.5).

The supergravity solution possesses an $\mathrm{SO}(5) \times \mathrm{U}(1)_R$ isometry, which reduces to $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)_R$ when we perform a $\mathbb{Z}_k$ quotient analogous to [4] with $k>1$. This is therefore the first example of a *non-toric* $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality. In fact there are very few examples of this kind, even in the more developed four-dimensional context. The singularity (1.1) is the $n=2$ member of a family of $\mathcal{A}_{n-1}$ four-fold singularities, defined by the hypersurface equations $X_n = \{z_0^n + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,, z_i \in \mathbb{C}\}$. Thus we are

[1] A different proposal was given in [11]. However, this was not based on Chern-Simons theory.

JHEP12(2009)017

naturally led to consider a family of Chern-Simons-quiver theories, labelled by n, whose Abelian classical moduli spaces are precisely these singularities. Here the $n = 1$ model is the ABJM theory of [4]. Naively, this suggests that each of these theories will have a large N gravity dual given by $\mathrm{AdS}_4 \times Y_n$, where Y_n is a Sasaki-Einstein manifold defined by $Y_n = X_n \cap S^9$. However, the results of [12] prove that for $n > 2$ these Sasaki-Einstein metrics *do not exist*. This means that the field theories we construct cannot[2] flow to dual conformal fixed points in the IR. We will review the argument for this in the course of the paper. Nevertheless, we can study these theories in the UV, and in particular we can, and will, discuss their string theory duals in terms of a slight generalization of the Type IIB Hanany-Witten brane configurations [14]. This will allow us to derive field theory dualities, in which the ranks of the gauge groups change, using the Hanany-Witten brane creation effect. We emphasize again that the AdS_4 Freund-Rubin solutions exist only in the case $n = 1$ (the ABJM theory) and $n = 2$.

One of the motivations for studying these models is that on the gravity side there exists a smooth[3] supersymmetric solution which approaches asymptotically the $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ background [15]. For $k = 1$ this solution is a warped product $\mathbb{R}^{1,2} \times T^*S^4$, where T^*S^4 denotes the cotangent bundle of S^4, and there is a self-dual G-flux through the S^4 zero-section. In fact, the deformed solution corresponds to *deforming* the hypersurface singularity by setting the right hand side of equation (1.1) to a non-zero value. This is a complex Calabi-Yau deformation, precisely analogous to the familiar deformation of the conifold in six dimensions. Indeed, superficially this solution looks like the M-theory version of the Type IIB solution of Klebanov-Strassler [16]. In the IR the two solutions are precisely analogous; however, in the UV they behave rather differently. In particular, the M-theory solution here is asymptotically $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$, without the logarithmic corrections which are a distinctive feature of the solutions of [16–18]. The topology of the solution at infinity can support only torsion G-flux, but a careful analysis reveals that in fact in the deformed solution this torsion flux is zero. Thus we are led to conjecture that the theory in the UV is the superconformal Chern-Simons-quiver theory above, with *equal* ranks of the two gauge groups. We will argue that this solution corresponds to an RG flow triggered by adding a supersymmetric mass term to the Lagrangian. This was already observed in [19], but we will here describe in more detail the deformation in terms of the superconformal Chern-Simons theory. In particular, we will see how the deformation of the field theory modifies the (classical) vacuum moduli space, precisely reproducing the deformation of the singularity (1.1).

The plan of the paper is as follows. In section 2 we introduce the Chern-Simons-quiver field theories: we compute their classical vacuum moduli spaces and discuss the relation to parent four-dimensional theories. In section 3 we discuss M-theory and Type IIA duals of these Chern-Simons theories. In section 4 we construct Hanany-Witten brane configurations in Type IIB string theory, and discuss a brane creation effect in these models. In section 5 we describe the deformed supergravity solution. In section 6 we identify this

[2] We note that it was suggested previously, incorrectly, that these singularities lead to AdS_4 holographic duals [13].

[3] The solution is completely smooth only for $k = 1$. For $k > 1$ there are orbifold singularities.

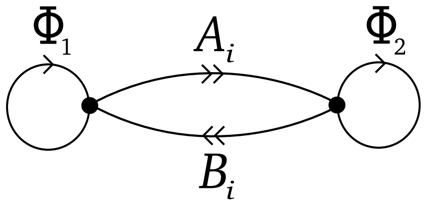

Figure 1. The $\mathcal{A}_1$ quiver.

deformed solution in the UV with a specific supersymmetric mass deformation of the field theory. Section 7 briefly concludes. We relegate some technical details, as well as a different Type IIA dual, to a number of appendices.

2 Field theories

We begin by describing a family of $d = 3$, $\mathcal{N} = 2$ Yang-Mills-Chern-Simons quiver theories. The family is labelled by a positive integer $n \in \mathbb{N}$, where the $n = 1$ theory is that of ABJM [4].

2.1 A family of $d = 3$, $\mathcal{N} = 2$ Chern-Simons-quiver theories

A $d = 3$, $\mathcal{N} = 2$ vector multiplet V consists of a gauge field $\mathscr{A}_\mu$, a scalar field σ, a two-component Dirac spinor χ, and another scalar field D, all transforming in the adjoint representation of the gauge group. This is simply the dimensional reduction of the usual $d = 4$, $\mathcal{N} = 1$ vector multiplet. For the theories of interest, we take the gauge group to be a product $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$. We will therefore have two vector multiplets V_I, $I = 1, 2$, with corresponding Yang-Mills gauge couplings g_I. To the usual $\mathcal{N} = 2$ Yang-Mills action, we may also add a Chern-Simons interaction. This requires specifying the Chern-Simons levels k_I, $I = 1, 2$, for the two gauge group factors. These are quantized: for $\mathrm{U}(N_I)$ or $\mathrm{SU}(N_I)$ gauge group $k_I \in \mathbb{Z}$ is an integer. In this paper we shall only consider the case that $k_1 = -k_2 \equiv k$; for $k_1 + k_2 \neq 0$ the dual string theory description will be in terms of massive Type IIA [20], which we do not wish to consider here.

The matter fields of an $\mathcal{N} = 2$ theory are described by chiral multiplets, a multiplet consisting of a complex scalar ϕ, a fermion ψ and an auxiliary scalar F, which may be in an arbitrary representation of the gauge group. For the theories of interest, we consider chiral fields A_i, $i = 1, 2$, transforming in the $\bar{\mathbf{N}}_\mathbf{1} \otimes \mathbf{N}_\mathbf{2}$ representation of $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$, and bifundamentals B_i, $i = 1, 2$, transforming in the conjugate $\mathbf{N}_\mathbf{1} \otimes \bar{\mathbf{N}}_\mathbf{2}$ representation. We also introduce chiral fields Φ_I, $I = 1, 2$, in the adjoint representation of $\mathrm{U}(N_I)$, respectively. This gauge and matter content is a quiver gauge theory, where the quiver is known as the $\mathcal{A}_1$ quiver. This is shown in figure 1.

The total Lagrangian then consists of the four terms (see e.g. [6, 21])

$$S = S_{\mathrm{YM}} + S_{\mathrm{CS}} + S_{\mathrm{matter}} + S_{\mathrm{potential}} \,, \tag{2.1}$$

where the bosonic parts of the Chern-Simons and matter Lagrangian are

$$S_{\rm CS} = \sum_{I=1}^{2} \frac{k_I}{4\pi} \int \mathrm{Tr} \left(\mathscr{A}_I \wedge \mathrm{d}\mathscr{A}_I + \frac{2}{3} \mathscr{A}_I \wedge \mathscr{A}_I \wedge \mathscr{A}_I + 2 D_I \sigma_I \right) , \tag{2.2}$$

$$S_{\rm matter} = \sum_a \int \mathrm{d}^3 x \mathscr{D}_\mu \bar{\phi}_a \mathscr{D}^\mu \phi_a - \bar{\phi}_a \sigma^2 \phi_a + \bar{\phi}_a D \phi_a \, , \tag{2.3}$$

respectively, where $\phi_a = (A_i, B_i, \Phi_I)$. In (2.3), the σ and D fields act in the appropriate representation on the ϕ_a — see [6, 21]. The Yang-Mills terms will, at low energies, be irrelevant. Finally, the F-term potential is

$$S_{\rm potential} = - \sum_a \int \mathrm{d}^3 x \left| \frac{\partial W}{\partial \phi_a} \right|^2 , \tag{2.4}$$

and we take the following superpotential:

$$W = \mathrm{Tr} \left[s \left((-1)^n \Phi_1^{n+1} + \Phi_2^{n+1} \right) + \Phi_2 (A_1 B_1 + A_2 B_2) + \Phi_1 (B_1 A_1 + B_2 A_2) \right] . \tag{2.5}$$

Here $n \in \mathbb{N}$ is a positive integer, and s is a complex coupling constant. The superpotential is manifestly invariant under an $\mathrm{SU}(2)_r$ flavour[4] symmetry under which the adjoints Φ_I are singlets and both pairs of bifundamentals A_i, B_i transform as doublets. There is also a $\mathbb{Z}_2^{\rm flip}$ symmetry which exchanges $\Phi_1 \leftrightarrow \Phi_2$, $A_i \leftrightarrow B_i$, $s \leftrightarrow (-1)^n s$.

The case $n = 1$ is special, since then the first two terms in (2.5) give a mass to the adjoint fields Φ_1, Φ_2. At low energy, we may therefore integrate out these fields. On setting $s = k/8\pi$, one recovers the ABJM theory with quartic superpotential [4]

$$W_{\rm ABJM} = \frac{4\pi}{k} (A_1 B_2 A_2 B_1 - A_1 B_1 A_2 B_2) . \tag{2.6}$$

This theory is in fact superconformal with enhanced manifest $\mathcal{N} = 6$ supersymmetry. We shall discuss the IR properties of the $n > 1$ theories after first discussing their vacuum moduli spaces.

2.2 Vacuum moduli spaces

We denote the ranks by $N_1 = N + l$, $N_2 = N$, and consider the vacuum moduli space of the theory $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$. In general there are six F-term equations derived from imposing vanishing of (2.4), which is $\mathrm{d}W = 0$:

$$\begin{aligned} B_i \Phi_2 + \Phi_1 B_i &= 0 \, , \\ \Phi_2 A_i + A_i \Phi_1 &= 0 \, , \\ s(n+1) \Phi_2^n + (A_1 B_1 + A_2 B_2) &= 0 \, , \\ s(-1)^n (n+1) \Phi_1^n + (B_1 A_1 + B_2 A_2) &= 0 \, . \end{aligned} \tag{2.7}$$

One must also impose the three-dimensional analogue of the D-term equations [6], and divide by the gauge symmetry.

[4] The reason for the subscript r will become apparent later. It is *not* to be confused with an R-symmetry.

It is easier to understand this moduli space in stages, starting with the Abelian theory with $k = 1$. In the $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge theory, as usual in quiver theories the diagonal $\mathrm{U}(1)$ decouples (no matter field is charged under it). Precisely as in the ABJM theory at Chern-Simons level $k = 1$, the anti-diagonal $\mathrm{U}(1)$, which we denote $\mathrm{U}(1)_b$, may be gauged away because of the Chern-Simons interaction. Thus the vacuum moduli space, in the Abelian case with $k = 1$, is described purely by the set of F-terms (2.7). The first four equations are reducible: either $\Phi_1 = -\Phi_2$, or else $A_i = B_j = 0$ for all i, j. In the latter case the last two equations imply $\Phi_1 = \Phi_2 = 0$, so this is not a separate branch. Thus $\Phi_1 = -\Phi_2$ holds in general, and we obtain the single equation for the moduli space

$$s(n+1)\Phi_2^n + A_1B_1 + A_2B_2 = 0 \,. \tag{2.8}$$

After the change of coordinates $z_1 = \frac{1}{2}(A_1 + B_1)$, $z_2 = \frac{\mathrm{i}}{2}(A_1 - B_1)$, $z_3 = \frac{1}{2}(A_2 + B_2)$, $z_4 = \frac{\mathrm{i}}{2}(A_2 - B_2)$, $z_0 = (s(n+1))^{\frac{1}{n}}\Phi_2$, this becomes simply

$$X_n \equiv \left\{ z_0^n + \sum_{a=1}^{4} z_a^2 = 0 \right\} \,. \tag{2.9}$$

For $n = 1$ this is indeed just $\mathbb{C}^4$, as one expects since this is the Abelian ABJM theory with $k = 1$, which corresponds to the theory on an M2-brane in flat spacetime. For $n > 1$, (2.9) instead describes an isolated four-fold hypersurface singularity, where the isolated singularity is at the origin $\{z_0 = z_1 = \cdots = z_4 = 0\}$. This is Calabi-Yau, in the sense that away from the singular point there is a global nowhere-zero holomorphic $(4,0)$-form. We denote the four-fold singularity by X, or X_n when we wish to emphasize the n-dependence. In particular, $X_1 \cong \mathbb{C}^4$. We shall study these varieties in more detail later.

The effect of changing the Chern-Simons levels to $(k, -k)$ leads to a discrete quotient of the above vacuum moduli space by $\mathbb{Z}_k \subset \mathrm{U}(1)_b$ [4, 6, 22]. Here by definition the charges of (A_1, A_2, B_1, B_2) under $\mathrm{U}(1)_b$ are $(1, 1, -1, -1)$, while the adjoints are uncharged. Thus for general k the Abelian vacuum moduli space is $X_n/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts freely away from the isolated singular point. Thus $X_n/\mathbb{Z}_k$ is also an isolated four-fold singularity.

Having understood the moduli space for the $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-k}$ theory, we may now turn to the general non-Abelian $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$ theory. The discussion here is similar to that for the ABJM theory in [4, 23]. In vacuum, Φ_1, σ_1 are $(N+l) \times (N+l)$ matrices (with σ_I Hermitian), Φ_2, σ_2 are $N \times N$ matrices, while the A_i and B_i are $N \times (N+l)$ and $(N+l) \times N$ matrices, respectively. Note that using the gauge symmetry one may always diagonalize the σ_I. The latter are fixed by the chiral field VEVs via three-dimensional analogues of the four-dimensional D-term equations [6], with the σ_I playing the role of moment map levels. If we take *all* matrices to be diagonal in the obvious $N \times N$ sub-blocks, so that the chiral fields take the form

$$\phi_a^{AB} = \delta^{AB}\phi_a^A \,, \qquad A, B = 1, \ldots, N \,, \tag{2.10}$$

with all other entries zero, then it is simple to see that the scalar potential is zero provided the ϕ_a^A, $A = 1, \ldots, N$, satisfy the Abelian equations (the F-terms $\Phi_1^A = -\Phi_2^A$, (2.8), and the D-term equations involving the σ_I^A). It is also straightforward to see from the D-term potential that for *generic* σ_I (meaning pairwise non-equal eigenvalues), all off-diagonal fluctuations about any vacuum in this space of vacua are massive, with the exception of fluctuations of Φ_1 in the $l\times l$ sub-block. The diagonal ansatz for the fields breaks the gauge symmetry to $\mathrm{U}(1)^N \times \mathrm{U}(l) \times \mathrm{U}(1)^N \times S_N$, i.e. we obtain precisely N copies of the Abelian $N = 1$ theory, where the permutation group S_N permutes the diagonal elements (it is the Weyl group of the diagonal $\mathrm{U}(N)$). We also obtain a $\mathrm{U}(l)_k$ Chern-Simons theory, as in [23], but for general n we also obtain a superpotential term Ψ^{n+1}, where Ψ is an adjoint under $\mathrm{U}(l)$ coming from the $l\times l$ sub-block of Φ_1. Classically this has a trivial moduli space, since the F-term gives $\Psi = 0$. Thus classically we obtain the symmetric product of N copies of the Abelian vacuum moduli space, i.e. $\mathrm{Sym}^N(X_n/\mathbb{Z}_k)$.

However, as for the ABJM theory, in the quantum theory this moduli space can be lifted. In particular, the $\mathrm{U}(l)_k$ Chern-Simons theory with an adjoint superpotential Ψ^{n+1} has been studied in the literature before — for a recent account, together with a D-brane engineering of this theory, see for example [24] and [25]. As reviewed in the latter reference, around equation (2.4), the above Chern-Simons theory has *no supersymmetric vacuum* unless $0 \leq l \leq nk$. This suggests that the above classical space of vacua is lifted unless this condition on l is obeyed. As we shall see later in the paper, this condition is also realized non-trivially in the M-theory dual, and leads to a 1-1 matching between the field theories $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$, with $0 \leq l < nk$, and the M-theory backgrounds we shall describe in section 3 (the theories with $l = 0$ and $l = nk$ will turn out to be dual to each other under a Seiberg-like duality that we derive using the Type IIB brane dual in section 4).

2.3 IR fixed points

As mentioned already, for $n = 1$ the fields Φ_1, Φ_2 are massive and on integrating these out we recover at low energies the ABJM theory. This has $\mathcal{N} = 6$ superconformal invariance for general $k \in \mathbb{Z}$. For $n > 1$ the IR dynamics is rather different. Anticipating much of the discussion that will follow later in section 3, we may use the AdS/CFT correspondence to conjecture that the theory with $n = 2$ and equal ranks $N_1 = N_2 = N$ flows to a strongly coupled $\mathcal{N} = 2$ superconformal fixed point in the IR. The reason for this is that in this case there exists a candidate gravity dual: an $\mathrm{AdS}_4 \times Y_2/\mathbb{Z}_k$ Freund-Rubin solution of eleven-dimensional supergravity, where Y_2 is a Sasaki-Einstein seven-manifold. More precisely, the four-fold hypersurface singularity X_2 admits a *conical* Calabi-Yau (Ricci-flat Kähler) metric, where the base of the cone is described by a homogeneous Sasaki-Einstein metric on Y_2 — we shall discuss this in detail in section 3. Notice that, since W has R-charge/scaling dimension precisely 2, all of the fields $\phi_a = (A_i, B_i, \Phi_I)$ must have R-charge/scaling dimension 2/3 at this fixed point, showing that it is strongly coupled. As we shall also see in section 3, more precisely we conjecture this fixed point with equal ranks N to be dual to the Freund-Rubin Sasaki-Einstein background with zero internal G-flux: as for the ABJM theory [23], more generally it is possible to turn on l units of discrete torsion G-flux, where in the gravity solution l is an integer mod nk, which is dual to changing the ranks to $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$, as discussed at the end of the previous subsection.

JHEP12(2009)017

On the other hand, it was shown in [12] that for $n > 2$ the natural candidate Sasaki-Einstein metrics do not actually exist; that is, the four-fold hypersurface singularities X_n, for $n > 2$, do not have Calabi-Yau cone metrics. This indicates that the corresponding field theories cannot flow to conformal fixed points dual to these geometries. Indeed, the field theory realization of this was also described in [12]: if the superpotential is (2.5) at the IR fixed point, then the gauge invariant chiral primary operators $\mathrm{Tr}\,\Phi_I$ have R-charge/scaling dimension $2/(n+1)$; but for $n > 2$ this violates the unitarity bound, which requires $\Delta \geq 1/2$, with equality only for a free field. It is therefore natural to conjecture that for $n > 2$ the higher order terms in Φ_I in (2.5) are irrelevant in the IR, and thus $s = 0$ at the IR fixed point. If this is the case, then all the theories with $n > 2$ flow to the *same* fixed point theory, namely the theory with $s = 0$.

Consider then setting $s = 0$ in W in (2.5). If we also set $k = 0$, so that there is no Chern-Simons interaction, this is precisely the $\mathcal{A}_1$ quiver gauge theory. For equal ranks $N_1 = N_2 = N$, the latter is well-known to be the low-energy effective theory on N D2-branes transverse to $\mathbb{R} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$; here $\mathbb{C}^2/\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts via $(z_1, z_2) \mapsto (-z_1, -z_2)$, is precisely the $\mathcal{A}_1$ singularity. The latter has an isolated singularity at the origin, where the N D2-branes are placed. This may be resolved by blowing up to $\mathcal{O}(-2) \to \mathbb{CP}^1$ (the Eguchi-Hanson manifold). If we wrap l space-filling D4-branes over the $\mathbb{CP}^1$ zero-section, the ranks are instead $N_1 = N + l$, $N_2 = N$. This theory has enhanced $\mathcal{N} = 4$ supersymmetry. If we now turn on the Chern-Simons coupling $k \neq 0$, the Abelian vacuum moduli space of the resulting theory is easily checked to be $\mathbb{C} \times \mathrm{Con}/\mathbb{Z}_k$, where $\mathrm{Con} = \{xy = uv\} \subset \mathbb{C}^4$ denotes the conifold three-fold singularity. Since this (non-isolated) four-fold singularity certainly admits a Calabi-Yau cone metric, this describes the candidate AdS dual to the IR fixed points of the theories with $n > 2$. It would be interesting to study this further.

2.4 Parent $d = 4$, $\mathcal{N} = 1$ theories and Laufer's resolution

As discussed in [6], the gauge group, matter content and superpotential of a $d = 3$, $\mathcal{N} = 2$ Chern-Simons matter theory also specify a $d = 4$, $\mathcal{N} = 1$ gauge theory — one takes the same Yang-Mills action, matter kinetic terms and superpotential interaction, now defined in $d = 4$, and simply discards the Chern-Simons level data (since the Chern-Simons interaction doesn't exist in four dimensions). This is commonly referred to as the "parent theory". The classical vacuum moduli space of this $d = 4$ parent theory is closely related to that of the $d = 3$ Chern-Simons theory [6]. The string theoretic relation between the two theories was recently elucidated in [8], and we shall make use of this correspondence later in the paper. The $d = 4$ parents of the above theories have been discussed extensively in the literature — in particular, see [26]. We are not interested in the four-dimensional theories directly; however, it will be useful to analyse their Abelian vacuum moduli spaces, and in particular the moduli spaces with a non-zero Fayet-Iliopoulos (FI) parameter turned on.

Compared to the $d = 3$ Chern-Simons matter theory, the only difference in constructing the Abelian vacuum moduli space of the $d = 4$ parent is that the $\mathrm{U}(1)_b$ gauge symmetry now acts faithfully on the vacuum moduli space. The analysis of the F-term equations is identical to that in section 2.2, and for the Abelian theory with equal ranks $N_1 = N_2 = 1$

we obtain the hypersurface equation (2.8). However, we must also impose the D-term

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \zeta \,, \tag{2.11}$$

and divide by $\mathrm{U}(1)_b$. Here we have introduced an FI parameter $\zeta \in \mathbb{R}$ for $\mathrm{U}(1)_b$.

Let us first set $\zeta = 0$. In this case, the combination of the D-term (2.11) and identifying by $\mathrm{U}(1)_b$ may be realized holomorphically by taking the holomorphic quotient by the complexification $\mathbb{C}^*_b$. The charges of (A_1, A_2, B_1, B_2) are $(1, 1, -1, -1)$, and thus the invariant functions on the quotient are spanned by $x = A_2B_2$, $y = A_1B_1$, $u = A_1B_2$, $v = A_2B_1$. These satisfy the single relation

$$xy = uv \,, \tag{2.12}$$

which is the conifold singularity. We must also impose the F-term (2.8), which setting $z_0 = (s(n+1))^{\frac{1}{n}}\Phi_2$, as before, reads

$$x + y + z_0^n = 0 \,. \tag{2.13}$$

Combining (2.13) with (2.12), and again changing variables $u = A_1B_2 = \mathrm{i}w_2 - w_3$, $v = A_2B_1 = \mathrm{i}w_2 + w_3$, $y = A_1B_1 = \mathrm{i}w_1 - w_0^n$, $z_0 = [s(n+1)]^{1/n}\Phi_2 = 2^{1/n}w_0$ gives the three-fold singularity

$$W_n^0 \equiv \left\{ w_0^{2n} + w_1^2 + w_2^2 + w_3^2 = 0 \right\} \,. \tag{2.14}$$

This is an isolated three-fold singularity, and is again Calabi-Yau in the sense that there is a holomorphic volume form on the complement of the singular point $\{w_0 = w_1 = w_2 = w_3 = 0\}$.

Taking the parameter $\zeta \neq 0$ in (2.11), one obtains a "small" resolution of the singularity W_n^0. It is small in the sense that the singular point is replaced by a one-dimensional (rather than two-dimensional) complex submanifold – specifically, a $\mathbb{CP}^1$. More precisely, for $\zeta > 0$ we obtain a resolution $W_n^\zeta \cong W_n^+$, where "$\cong$" means biholomorphic, while for $\zeta < 0$ we obtain a resolution $W_n^\zeta \cong W_n^-$. In both cases the "exceptional" $\mathbb{CP}^1$ has size $|\zeta|$ in the induced Kähler metric. Indeed, any Kähler metric on W_n^ζ will have a Kähler class in $H^2(W_n^\zeta, \mathbb{R}) \cong \mathbb{R}$, and we regard ζ as specifying this Kähler class. Both resolutions are also Calabi-Yau, in the sense that there is a holomorphic volume form, and are thus "crepant".

More on W_n^ζ

The end of this section is more technical, and may be skipped on a first reading.

To see why W_n^ζ takes the form described above, recall that the F-term equation (2.8) describes the moduli space in terms of coordinates $(A_1, A_2, B_1, B_2, \Phi_2)$ on $\mathbb{C}^5$. Imposing the D-term (2.11) and dividing by $\mathrm{U}(1)_b$ then gives $\mathrm{Con}_\zeta \times \mathbb{C}$, where the resolved conifold Con_ζ is obtained from the quotient of the (A_1, A_2, B_1, B_2) coordinates, while the VEV of Φ_2 is a coordinate on $\mathbb{C}$. In particular, $\zeta > 0$ and $\zeta < 0$ are related by the conifold flop transition. The exceptional $\mathbb{CP}^1$ in the resolved conifold is at $B_1 = B_2 = 0$ for $\zeta > 0$, and $A_1 = A_2 = 0$ for $\zeta < 0$, respectively. The three-fold W_n^ζ is then embedded in

$\mathrm{Con}_\zeta \times \mathbb{C}$ via (2.8). We may also realize the D-term mod $\mathrm{U}(1)_b$ as a $\mathbb{C}_b^*$ quotient. Strictly speaking, this is a geometric invariant theory quotient, and for $\zeta > 0$ we need to remove the (unstable) points $\{A_1 = A_2 = 0\}$, while for $\zeta < 0$ we instead remove $\{B_1 = B_2 = 0\}$. Without loss of generality we henceforth take $\zeta > 0$ (as $\zeta < 0$ is just related by a flop), and thus remove $\{A_1 = A_2 = 0\}$ from $\mathbb{C}^4$, spanned by (A_1, A_2, B_1, B_2). Define coordinate patches $U_i = \{A_i \neq 0\} \subset \mathbb{C}^4$, $i = 1, 2$. These will cover the manifold, as A_1 and A_2 cannot both be zero. On U_1 the invariant functions under $\mathbb{C}_b^*$ are spanned by $x = A_2 B_2$, $y = A_1 B_1$, $u = A_1 B_2$, $v = A_2 B_1$, $\xi = A_2/A_1$, while on U_2 the invariant functions are the same x, y, u, v, but instead $\mu = A_1/A_2$. We then have the relations

$$
\begin{aligned}
x &= u\xi\,, \qquad & v &= y\xi\,, \qquad & &\text{on} \quad U_1\,, \\
u &= x\mu\,, \qquad & y &= v\mu\,, \qquad & &\text{on} \quad U_2\,.
\end{aligned}
\tag{2.15}
$$

It follows that we may coordinatize U_1 by (u, y, ξ) and U_2 by (x, v, μ), with transition functions $(x, v, \mu) = (u\xi, y\xi, 1/\xi)$ on the overlap $U_1 \cap U_2$. This shows explicitly the resolved conifold as $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, where ξ and μ are coordinates on the two patches of the Riemann sphere $\mathbb{CP}^1$, with $\mu = 1/\xi$ on the overlap. The poles of the sphere are thus $\mu = 0$ and $\xi = 0$.

The three-fold $W_n^+ \cong W_{\zeta>0}$ is embedded as a complex hypersurface in the resolved conifold times $\mathbb{C}$. We thus introduce patches H_1, with coordinates (u, y, ξ, Z_1), and H_2, with coordinates (x, v, μ, Z_2), where $Z_1 = Z_2 = \Phi_2$ is the coordinate on $\mathbb{C}$. The embedding equation (2.8) is then simply

$$
\begin{aligned}
y &= -u\xi - Z_1^n \qquad & &\text{on} \quad H_1\,, \\
x &= -v\mu - Z_2^n \qquad & &\text{on} \quad H_2\,.
\end{aligned}
\tag{2.16}
$$

We may thus eliminate x and y and coordinatize H_1 by (u, ξ, Z_1) and H_2 by (v, μ, Z_2), with transition functions $(v, \mu, Z_2) = (-\xi Z_1^n - \xi^2 u, 1/\xi, Z_1)$ on the overlap $H_1 \cap H_2$. This is precisely the description of the small crepant resolution W_n^+ of W_n^0 given by Laufer [27]. One sees explicitly the exceptional $\mathbb{CP}^1$ with coordinates ξ, μ, and $\mu = 1/\xi$ on the overlap. One also sees that for $n = 1$ the normal bundle of $\mathbb{CP}^1$ inside W_n^+ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, while for all $n \geq 2$ the normal bundle is instead $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{CP}^1$.

3 M-theory and type IIA duals

In this section we discuss M-theory and Type IIA duals to the Chern-Simons-quiver theories of section 2.1. We have already shown that the vacuum moduli space of the $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$ theory is $\mathrm{Sym}^N X_n/\mathbb{Z}_k$, and this suggests a dual M-theory interpretation in terms of N M2-branes probing the four-fold singularity $X_n/\mathbb{Z}_k$. As in [23], we show that the integer l, which is constrained to lie in the interval $0 \leq l \leq nk$ in the field theory, may be identified with turning on l units of torsion G-flux in the M-theory background. On the gravity side, l is defined only modulo nk — we will have to wait until section 4 to see why the $l = 0$ field theory is dual to the $l = nk$ theory.

As already mentioned, only for $n = 1$, $n = 2$ do the four-fold singularities X_n have Ricci-flat Kähler *cone* metrics, implying that only in this case do the conformal fixed points of the Chern-Simons-quiver theories have AdS duals of this type; we conjectured that for all $n > 2$ the theories flow to the *same* fixed point theory in the IR, and that this has a different AdS dual description where the Sasaki-Einstein seven-space is the singular link of $\mathbb{C} \times \mathrm{Con}/\mathbb{Z}_k$. Although we are interested primarily in the case $n = 2$, we retain n throughout this section and study M-theory on $\mathrm{AdS}_4 \times Y_n/\mathbb{Z}_k$, where Y_n is the link of the singularity X_n. We stress again, however, that the AdS_4 solutions of this type exist *only* for $n = 1$, $n = 2$.

3.1 M-theory duals

The discussion of section 2.2 suggests that the Chern-Simons quivers of section 2.1 should have M-theory duals in terms of M2-branes placed at the four-fold singularities $X_n/\mathbb{Z}_k$ (2.9). Thus it is natural to conjecture that the IR fixed points of the Chern-Simons quivers, for $n = 1$, $n = 2$, are SCFTs dual to the gravity backgrounds $\mathrm{AdS}_4 \times Y_n/\mathbb{Z}_k$, where Y_n is the base of the cone X_n, equipped with a Sasaki-Einstein metric. The case $n = 1$ is just the round metric on $Y_1 = S^7$, which is the ABJM model. The case $n = 2$ leads instead to $Y_2 = V_{5,2}$, where $V_{5,2}$ has a homogeneous Sasaki-Einstein metric that we discuss below.

Consider the complex cone X_n defined in (2.9). We may define the compact seven-manifold Y_n via

$$Y_n \equiv X_n \cap S^9 \,, \tag{3.1}$$

where $S^9 = \{\sum_{i=0}^4 |z_i|^2 = 1\} \subset \mathbb{C}^5$. For $n = 1$ this is simply $Y_1 = S^7$, so we focus on describing Y_2. In this case X_2 is a complex quadric, and the vector action of $\mathrm{SO}(5)$ on the coordinates z_i acts transitively on the seven-manifold Y_2, and thus $Y_2 = V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$ is a coset space. X_2 is also invariant under the rescaling $z_i \mapsto \lambda z_i$, for $\lambda \in \mathbb{C}^*$, and the quotient $B^6 \equiv (X_2 \setminus \{0\})/\mathbb{C}^*$ is a compact complex manifold of complex dimension three. Equivalently, this may be defined as $B^6 = V_{5,2}/\mathrm{U}(1)_R$, where $\mathrm{U}(1)_R$ acts on the z_i with charge 1, and thus $B^6 \cong \mathrm{Gr}_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3) \times \mathrm{SO}(2)$ is also a coset space. The space $\mathrm{Gr}_{5,2}$ is the Grassmanian of two-planes in $\mathbb{R}^5$.

There is an explicit homogeneous Sasaki-Einstein metric on $Y_2 = V_{5,2}$, so that the quadric singularity X_2 has a Ricci-flat Kähler cone metric. The Reeb $\mathrm{U}(1)$ action is precisely the action by $\mathrm{U}(1)_R \subset \mathbb{C}^*$ above; thus $V_{5,2}$ is a regular Sasaki-Einstein manifold and the quotient $\mathrm{Gr}_{5,2}$ is a homogeneous Kähler-Einstein manifold. The Sasaki-Einstein metric on $V_{5,2}$ may be written explicitly in suitable coordinates [28]

$$\mathrm{d}s^2(V_{5,2}) = \frac{9}{16}\left[\mathrm{d}\psi + \frac{1}{2}\cos\alpha(\mathrm{d}\beta - \cos\theta_1\mathrm{d}\phi_1 - \cos\theta_2\mathrm{d}\phi_2)\right]^2 + \mathrm{d}s^2(\mathrm{Gr}_{5,2})\,, \tag{3.2}$$

where

$$\mathrm{d}s^2(\mathrm{Gr}_{5,2}) = \frac{3}{32}\Big[4\mathrm{d}\alpha^2 + \sin^2\alpha(\mathrm{d}\beta - \cos\theta_1\mathrm{d}\phi_1 - \cos\theta_2\mathrm{d}\phi_2)^2 \\ +(1+\cos^2\alpha)(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2) \\ +2\sin^2\alpha\cos\beta\sin\theta_1\sin\theta_2\mathrm{d}\phi_1 d\phi_2 - 2\sin^2\alpha\cos\beta\mathrm{d}\theta_1\mathrm{d}\theta_2 \\ +2\sin^2\alpha\sin\beta(\sin\theta_2\mathrm{d}\phi_2\mathrm{d}\theta_1 + \sin\theta_1\mathrm{d}\phi_1\mathrm{d}\theta_2)\Big] \quad (3.3)$$

is the homogeneous Kähler-Einstein metric on $B^6 = \mathrm{Gr}_{5,2}$. The ranges of the coordinates are

$$0 \le \theta_i \le \pi\,, \quad 0 \le \phi_i < 2\pi\,, \quad 0 \le \psi < 2\pi\,, \quad 0 \le \alpha \le \frac{\pi}{2}\,, \quad 0 \le \beta < 4\pi\ . \quad (3.4)$$

The volume of the Sasaki-Einstein metric on $V_{5,2}$ is [28]

$$\mathrm{vol}(V_{5,2}) = \frac{27}{128}\pi^4\ . \quad (3.5)$$

Notice the isometry group of the homogeneous metric on $V_{5,2}$ is $\mathrm{SO}(5)\times\mathrm{U}(1)_R$, and thus in particular this is a *non toric* manifold.

Thus for $n=1$, $n=2$ we have supersymmetric Freund-Rubin backgrounds of eleven-dimensional supergravity of the type $\mathrm{AdS}_4\times Y_n$, with $Y_1 = S^7$ and $Y_2 = V_{5,2}$. The metric and G-field take the form[5]

$$\mathrm{d}s^2 = R^2\left(\frac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(Y_n)\right)\,, \\ G = \frac{3}{8}R^3\mathrm{dvol}(\mathrm{AdS}_4)\ . \quad (3.6)$$

The AdS_4 radius R is determined by the quantization of the G-flux

$$N = \frac{1}{(2\pi l_p)^6}\int_{Y_n} *G\,, \quad (3.7)$$

where l_p is the eleven-dimensional Planck length, given by

$$R^6 = \frac{(2\pi l_p)^6 N}{6\mathrm{vol}(Y_n)}\ . \quad (3.8)$$

We also note that $\mathrm{vol}(Y_1 = S^7) = \pi^4/3$.

Recall that in section 2 we introduced an action by the global symmetry group $\mathrm{U}(1)_b$. Writing the complex cone as $X_n = \{z_0^n + A_1B_1 + A_2B_2 = 0\}$, the $\mathrm{U}(1)_b$ symmetry acts on $(z_0, A_1, A_2, B_1, B_2)$ with charges $(0,1,1,-1,-1)$. This also acts on the base Y_n defined in (3.1), and it is easy to see that this is a *free* action, i.e. there are no fixed points on Y_n. For both $n=1$, $n=2$, $\mathrm{U}(1)_b$ acts isometrically on the Sasaki-Einstein metrics. In particular, for $n=2$ this embeds into the isometry group as $\mathrm{U}(1)_b \cong \mathrm{SO}(2)_{\mathrm{diagonal}} \subset \mathrm{SO}(4) \subset \mathrm{SO}(5)$. This is a non-R isometry, and so preserves the Killing spinors on $Y_2 = V_{5,2}$. We may

[5]The Einstein metrics on AdS_4 and Y_n obey $\mathrm{Ric}_{\mathrm{AdS}_4} = -3g_{\mathrm{AdS}_4}$, $\mathrm{Ric}_{Y_n} = 6g_{Y_n}$, respectively.

thus take a quotient of $V_{5,2}$ by $\mathbb{Z}_k \subset \mathrm{U}(1)_b$ to obtain a Sasaki-Einstein manifold $V_{5,2}/\mathbb{Z}_k$ with $\pi_1(V_{5,2}/\mathbb{Z}_k) \cong \mathbb{Z}_k$. Since $\mathrm{SO}(4) \cong (\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r)/\mathbb{Z}_2$, the diagonal SO(2) in SO(4) is $\mathrm{U}(1)_b \cong \mathrm{U}(1)_l \subset \mathrm{SU}(2)_l$. Thus the isometry group of the quotient space $V_{5,2}/\mathbb{Z}_k$ is $\mathrm{SU}(2)_r \times \mathrm{U}(1)_b \times \mathrm{U}(1)_R$. This is the manifest global symmetry in the Chern-Simons-quiver theories.

We conjecture that the Chern-Simons-quiver theory $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, with matter content given by the quiver in figure 1 and superpotential interaction (2.5) with $n = 2$, flows to a conformal fixed point in the IR, and is dual to the above $\mathrm{AdS}_4 \times Y_2/\mathbb{Z}_k$ M-theory background. As evidence for this, we have shown that the moduli space of the field theory agrees with the moduli space of N M2-branes probing the cone geometry, and that the isometry group of the AdS_4 solution precisely matches the global symmetries[6] of the field theory. Later in sections 3.3 and 3.4 we shall present a matching of various gauge invariant chiral primary operators to supergravity multiplets and certain supersymmetric wrapped D-branes, respectively, as further evidence. In section 4 we will also present a Type IIB brane construction.

Let us now discuss turning on a torsion C-field, corresponding to the addition of fractional branes [23]. As shown in appendix A, in general we have $H^4(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$, and thus we may turn on a torsion[7] G-field, i.e. a flat, but topologically non-trivial, G-flux. Each different choice of such G-flux will lead to a physically distinct M-theory background. We may equivalently describe this as a (discrete) holonomy for the three-form potential C through the Poincaré dual generator Σ^3 of $H_3(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$. Thus

$$\frac{1}{(2\pi l_p)^3} \int_{\Sigma^3} C = \frac{l}{nk} \mod 1 . \tag{3.9}$$

Since the physical gauge invariant object is a holonomy, the integer l above is only defined modulo nk. Equivalently, this labels the G-flux $[G] = l \in H^4(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$. For each choice of l with $0 \leq l < nk$ we therefore have a 1-1 matching of the M-theory backgrounds to the field theories with gauge groups $\mathrm{U}(N + l)_k \times \mathrm{U}(N)_{-k}$. We shall present further evidence for matching the G-flux to the ranks in this way from the Type IIA dual in section 3.5.

3.2 Type IIA duals

When $k^5 \gg N \gg k$ the radius of the $\mathrm{U}(1)_b$ circle becomes small and a better description is obtained by reducing the background along $\mathrm{U}(1)_b$ to a Type IIA configuration. Since $\mathrm{U}(1)_b$ acts freely on Y_n, we may define quite generally $M_n = Y_n/\mathrm{U}(1)_b$, which is a smooth six-manifold. For $n = 1$ this gives $M_1 = \mathbb{CP}^3$, while for $n > 1$ the manifold M_n has the same cohomology groups as $\mathbb{CP}^3$, but a cohomology ring that depends on n, as shown in

[6] As often happens in $\mathrm{AdS}_4/\mathrm{CFT}_3$, for $k = 1$ the isometry group is enhanced. In particular we have $\mathrm{SO}(5) \times \mathrm{U}(1)_R$ symmetry, rather than the $\mathrm{SU}(2)_r \times \mathrm{U}(1)_b \times \mathrm{U}(1)_R$ symmetry valid for $k > 1$. This former symmetry is not manifest in the UV Lagrangian.

[7] It is important here that the G-flux is classified topologically by $H^4(Y, \mathbb{Z})$, which is true only if the membrane anomaly is zero [29]. In fact the membrane anomaly always vanishes on any oriented spin seven-manifold.

appendix A. For $n = 2$, $\mathrm{U}(1)_b$ is a non-R symmetry, and therefore all supersymmetries are preserved in the quotient $V_{5,2}/\mathrm{U}(1)_b = M_2$. On the other hand, the Type IIA reduction of $\mathcal{N} = 2$ Freund-Rubin backgrounds along the R-symmetry (Reeb vector) direction breaks supersymmetry [30]. In particular, we stress that M_2 is *different* from the Kähler-Einstein six-manifold $\mathrm{Gr}_{5,2} = V_{5,2}/\mathrm{U}(1)_R$ introduced in section 3.1. These types of reduction were discussed in [31], and we now recall their essential features.

To perform the reduction we write the Sasaki-Einstein metric on $Y_n/\mathbb{Z}_k$ as

$$\mathrm{d}s^2(Y_n/\mathbb{Z}_k) = \mathrm{d}s^2(M_n) + \frac{w}{k^2}(\mathrm{d}\gamma + kP)^2 \,, \tag{3.10}$$

where γ has 2π period. We then obtain the following Type IIA string-frame metric and fields

$$\mathrm{d}s^2_{\mathrm{st}} \;=\; \sqrt{w}\,\frac{R^3}{k}\left(\frac{1}{4}\mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(M_n)\right) \,, \tag{3.11}$$

$$\mathrm{e}^{2\Phi} \;=\; \frac{R^3}{k^3}w^{3/2}\,, \qquad F_4 \;=\; \frac{3}{8}R^3\mathrm{dvol}(\mathrm{AdS}_4)\,, \qquad F_2 \;=\; kl_sg_s\mathrm{d}P\,, \tag{3.12}$$

where w is a nowhere-zero bounded function on M_n (since $\mathrm{U}(1)_b$ acts freely). The RR two-form flux has quantized periods, namely

$$\frac{1}{2\pi l_s g_s}\int_{\Sigma^2} F_2 \;=\; k \;. \tag{3.13}$$

Here $\Sigma^2 \subset M_n$ is the generator[8] of $H_2(M_n, \mathbb{Z}) \cong \mathbb{Z}$. Of course, these supergravity solutions exist only for $n = 1$, $n = 2$. In the latter case, then more precisely in terms of the coordinates in (3.2), (3.3) we have that $\gamma = \phi_2$ and

$$w = \frac{3}{32}\left[1 + \frac{1}{2}\cos^2\alpha(1 + \sin^2\theta_2)\right] \;. \tag{3.14}$$

The torsion C-field reduces to a flat NS B_2-field in Type IIA [23] via

$$C = A_3 + B_2 \wedge \mathrm{d}\psi \;. \tag{3.15}$$

Here A_3 denotes the RR three-form potential, while ψ parametrizes the M-theory circle with period $2\pi l_s g_s$, where recall that $l_p = l_s g_s^{1/3}$ is the eleven-dimensional Planck length. Denoting with $\Omega_2 = [\mathrm{d}P/2\pi]$ the generator of $H^2(M_n, \mathbb{Z}) \cong \mathbb{Z}$, we then have[9]

$$B_2 = (2\pi l_s)^2 \frac{l}{kn}\Omega_2 \;. \tag{3.16}$$

The period of B_2 through Σ^2 is hence

$$b \equiv \frac{1}{(2\pi l_s)^2}\int_{\Sigma^2} B_2 = \frac{l}{kn} \mod 1 \;. \tag{3.17}$$

[8] A detailed discussion of the topology of M_n is contained in appendix A.

[9] The authors of [32] argue, for the ABJM theory $n = 1$, that there is a shift in this B_2-field period by $1/2$ (in units of $(2\pi l_s)^2$). Notice that, ordinarily, the B_2-field period through Σ^2 would be a modulus, able to take any value in S^1 (after taking account of large gauge transformations). Since this does not affect our discussion, we shall not study this further here.

Again, as for the C-field period (3.9) through Σ^3, this is only defined modulo 1. In Type IIA, this is because large gauge transformations of the B_2-field change the period b by an integer.

3.3 Chiral primaries and their dual supergravity multiplets

We now turn to a discussion of the chiral primary operators of the $\mathcal{N}=2$ gauge theory with $n=2$, and how they are realized in the gravity dual. In the field theory we can construct chiral primary operators by taking appropriately symmetrized gauge-invariant traces of products of fields. These operators may be denoted very schematically as $\mathrm{Tr}\,[\Phi^{n_1}(AB)^{n_2}]$. They are invariant under $\mathrm{U}(1)_b$, and their dimension at the $n=2$ IR fixed point is $\Delta = 2/3\cdot(n_1+2n_2)$. However, because of the presence of monopole operators in three dimensions, these do not exhaust the list of all chiral primaries [4]. The monopole operator with a single unit of magnetic flux in the diagonal U(1) transforms in the $(\mathrm{Sym}^k(\mathbf{N}_1), \mathrm{Sym}^k(\bar{\mathbf{N}}_2))$ representation of the gauge group, and following [4] we may denote it as $\mathrm{e}^{\mathrm{i}\tau}$. Using this we can construct generalized gauge-invariant traces as

$$\mathrm{Tr}\,[\Phi^{n_1}(AB)^{n_2}A^{m_1 k}B^{m_2 k}\mathrm{e}^{\mathrm{i}(m_1-m_2)\tau}]\,, \quad n_i, m_i \in \mathbb{N}\,. \tag{3.18}$$

It is currently not known how to compute the dimensions of monopole operators in strongly coupled $\mathcal{N}=2$ Chern-Simons theories [33]. However, it is plausible that in the present case, as conjectured for the ABJM theory [4], their scaling dimension is zero. Assuming this, the dimensions of the operators (3.18) are then

$$\Delta = \frac{2}{3}[n_1 + 2n_2 + (m_1+m_2)k]\,. \tag{3.19}$$

These operators may be matched to a tower of states in the Kaluza-Klein spectrum on $V_{5,2}$ derived in [11]. Consider first setting $k=1$. The spectrum is arranged into supermultiplets, labelled by representations of $Osp(4|2)\times \mathrm{SO}(5)\times \mathrm{U}(1)_R$. When the corresponding dimensions of dual operators are rational, the multiplets undergo shortening conditions [34]. In particular, we see from table 6 of [11] that a certain vector multiplet ("Vector Multiplet II") becomes a short *chiral multiplet*, with components denoted as $(S/\Sigma, \lambda_L, \pi)$. These have spins $(0^+, 1/2, 0^-)$, respectively, and dimensions $(\Delta, \Delta+1/2, \Delta+1)$, with

$$\Delta = \frac{2}{3}m\,, \qquad m = 1,2,\ldots\,. \tag{3.20}$$

The lowest component fields then match the operators (3.18) with $m = n_1+2n_2+m_1+m_2$.

For $k>1$ only a subsector of these states survive the $\mathbb{Z}_k$ projection.[10] This is most easily seen using the equivalence of chiral primary harmonics on $V_{5,2}$ to *holomorphic functions* on the Calabi-Yau cone singularity X_2 [12]. These can be expanded in monomials of the form $\prod_{i=0}^{4} z_i^{s_i}$, for $s_i \in \mathbb{N}$. Using the results of [12] (see equation (3.22) of this reference)

[10] The representations that survive the $\mathbb{Z}_k$ projection are the singlets in the decomposition of $[m,0]$ under $\mathrm{SO}(5) \to \mathrm{SU}(2)_r \times \mathrm{U}(1)_b$.

we determine that the R-charges associated to the coordinates[11] z_i are all equal to 2/3, which of course agrees with (3.20). When $k > 1$ it is convenient to change coordinates and write the singularity as

$$z_0^2 + A_1 B_1 + A_2 B_2 \;=\; 0\,, \tag{3.21}$$

which diagonalizes the action of $\mathbb{Z}_k \subset \mathrm{U}(1)_b$. Recall that under $\mathrm{U}(1)_b$ these coordinates have charges $(0, 1, 1, -1, -1)$, respectively. Thus for $k > 1$ a general holomorphic function may be expanded in monomials of the form

$$z_0^{n_1} A^{p_1} B^{p_2}\,, \qquad p_1 - p_2 = 0 \mod k\,, \quad p_i \in \mathbb{N}\,. \tag{3.22}$$

These of course match precisley with the operators (3.18), where $p_1 = n_2 + m_1 k$, $p_2 = n_2 + m_2 k$.

For later purposes it will be useful to discuss the structure of the chiral multiplets on the gravity side in a little more detail. The lowest bosonic components S/Σ arise from a linear combination of metric modes and C-field modes in AdS_4. The top bosonic components π come purely from C-field modes in the internal directions, namely from certain massive harmonic three-forms on $Y = V_{5,2}$ — see table 1 of [11].

In the field theory, a chiral superfield may be written in superspace notation as $\Phi = \phi + \theta\psi + \theta^2 F$. The component fields have R-charges $(\Delta, \Delta - 1, \Delta - 2)$ and scaling dimensions $(\Delta, \Delta + 1/2, \Delta + 1)$, respectively. Then the bosonic physical degrees of freedom of a chiral operator of the form $\mathrm{Tr}\,\Phi^m$ are a scalar ϕ^m with dimension $m\Delta$, and a pseudoscalar $\psi^\alpha \psi_\alpha \phi^{m-2}$ with dimension $m\Delta + 1$. In the gravity dual, these are dual to the scalar modes S/Σ and the pseudoscalar modes π, respectively.

3.4 Baryon-like operators and wrapped branes

In this section we briefly discuss M5-branes wrapped on certain supersymmetric submanifolds in $Y_n/\mathbb{Z}_k$, and their Type IIA incarnation as D4-branes wrapped on submanifolds in M_n. These correspond to certain "baryonic" (i.e. determinant-like) operators in the field theories.

A full analysis of the spectrum of baryon-type operators is beyond the scope of this paper. However, we may provide further evidence for the proposed duality by analysing a certain simple set of operators. Thus, for the adjoint fields Φ_I we may consider the gauge-invariants $\det \Phi_I$, $I = 1, 2$. Notice that Φ_1 is an $(N + l) \times (N + l)$ matrix, while Φ_2 is $N \times N$. We may also define the (in general *non*-gauge-invariant) operators

$$\begin{aligned}
\mathscr{A}_i^{\gamma_1 \cdots \gamma_l} &\equiv \frac{1}{N!} \epsilon_{\alpha_1 \cdots \alpha_N} A_{i\,\beta_1}^{\alpha_1} \cdots A_{i\,\beta_N}^{\alpha_N} \epsilon^{\beta_1 \cdots \beta_N \gamma_1 \cdots \gamma_l}\,, \\
\mathscr{B}_{i\,\gamma_1 \cdots \gamma_l} &\equiv \frac{1}{N!} \epsilon^{\alpha_1 \cdots \alpha_N} B_{i\,\alpha_1}^{\beta_1} \cdots B_{i\,\alpha_N}^{\beta_N} \epsilon_{\beta_1 \cdots \beta_N \gamma_1 \cdots \gamma_l}\,.
\end{aligned} \tag{3.23}$$

[11] For general n, the would-be R-charges are $n/(n+1)$ for the coordinates $z_1, \ldots z_4$ and $2/(n+1)$ for the coordinate z_0. Therefore for $n > 3$ the latter violates the unitarity bound $\Delta \geq 1/2$, which geometrically is the Lichnerowicz bound. For $n = 3$ it saturates this bound, but one can still argue that the corresponding Sasaki-Einstein metric on Y_3 does not exist [12].

Here $\mathscr{A}_i$ lives in $\Lambda^l\overline{(\mathbf{N+l})}$, the lth antisymmetric product of the anti-fundamental representation of $\mathrm{U}(N+l)$, while $\mathscr{B}_i$ lives in $\Lambda^l(\mathbf{N+l})$ [35]. These are gauge-invariant only for $l=0$, but even in this case one needs to insert an appropriate monopole operator (see [33, 36] for a recent discussion of these operators); we will not study this here. For $l>0$, one can obtain gauge-invariant operators by, for example, taking $(N+l)$ copies of $\mathscr{A}_i$ and then contracting with l epsilon symbols for $\mathrm{U}(N+l)$ (with appropriate monopole operators). This situation is clearly much more complicated than it is for D3-branes in Type IIB string theory, and deserves further study. However, as for the ABJM theory, the operators (3.23) can still be matched to wrapped branes in the gravity dual, as we shall explain.

In M-theory we may associate these types of operators to M5-branes wrapping supersymmetric submanifolds. More precisely, these are the boundaries of divisors in the Calabi-Yau cone — see, e.g., the first reference in [9]. Given the discussion of the Abelian moduli space in section 2.2, we may associate the operators $\det\Phi_I$ with the divisor $\{z_0=0\}$ in the Calabi-Yau cone, while $\mathscr{A}_1$ is associated to $\{z_1=\mathrm{i}z_2\}$, $\mathscr{A}_2$ to $\{z_3=\mathrm{i}z_4\}$, $\mathscr{B}_1$ to $\{z_1=-\mathrm{i}z_2\}$, and $\mathscr{B}_2$ to $\{z_3=-\mathrm{i}z_4\}$. This follows by noting that, in the Abelian theory, the operators may be regarded as sections of line bundles over the Abelian vacuum moduli space; the divisors we have written are then the zeros of these sections.

Let us consider first the adjoints. Setting $z_0=0$ in X_n gives $\{z_1^2+z_2^2+z_3^2+z_4^2=0\}$, which is a copy of the conifold singularity. Thus the boundary $\Sigma_n^{(0)}$ of this divisor is a copy of $T^{1,1}$, for all n. Taking the $\mathbb{Z}_k$ quotient, one obtains instead $\Sigma_n^{(0)}/\mathbb{Z}_k=T^{1,1}/\mathbb{Z}_k$, where recall that $\mathbb{Z}_k$ is embedded in the diagonal SO(2) in SO(4). For the main case of interest, $n=2$, this can be seen explicitly in the polar coordinates of section 3.1: the five-dimensional submanifold $\Sigma_2^{(0)}$ corresponds to setting $\alpha=\beta=0$, and its volume is $\mathrm{vol}(\Sigma_2^{(0)})=(3\pi)^3/2^5$. We may also compute this volume using the results of [12, 28]. This gives the general result

$$\mathrm{vol}(\Sigma_n^{(0)})=\frac{(n+1)^3\pi^3}{4n^3}\ . \tag{3.24}$$

This is the volume of the submanifold induced by *any* Sasakian metric on Y_n with Reeb vector field weights $(4/(n+1),2n/(n+1),2n/(n+1),2n/(n+1),2n/(n+1))$. The latter are normalized so that the holomorphic $(4,0)$-form on the cone has charge 4. Similarly, one can compute

$$\mathrm{vol}(Y_n)=\frac{(n+1)^4\pi^4}{48n^3}\ . \tag{3.25}$$

This is then the volume of a Sasaki-Einstein metric on Y_n *if it exists*, which is true only for $n=1$, $n=2$. Using the formula for the dimension of the dual operator [37]

$$\Delta\ =\ \frac{N}{6}\frac{\pi\mathrm{vol}(\Sigma)}{\mathrm{vol}(Y)}\,, \tag{3.26}$$

we obtain in general $\Delta[\det\Phi_I]=2N/(n+1)$. Notice here that, since $\Sigma_n^{(0)}$ is invariant under $\mathrm{U}(1)_b$, after taking the $\mathbb{Z}_k$ quotient the dependence on k in the numerator and denominator in (3.26) cancel. This result then matches with the conformal dimensions of the adjoints computed from the constraint that the superpotential has scaling dimension 2.

However, the above discussion overlooks an important subtlety: we have two operators $\det\Phi_1$, $\det\Phi_2$, but only one divisor. Moreover, in the case of unequal ranks, $\mathrm{U}(N+l)_k \times \mathrm{U}(N)_{-k}$, one expects $\det\Phi_1$ to have dimension $\Delta \propto N+l$, while $\det\Phi_2$ should have dimension $\Delta \propto N$. In the case of D3-branes wrapping supersymmetric three-submanifolds in Sasaki-Einstein five-manifolds, there can also be multiple baryonic operators mapping to the same divisor: they are distinguished [38] physically in the gravity dual by having different flat worldvolume connections on the wrapped D3-branes. Here we have a wrapped M5-brane, and thus one expects the self-dual two-form on its worldvolume to play a similar role. Notice also that in general in the conformal dimension formula (3.26) one expects the on-shell M5-brane worldvolume action to appear in the numerator. In general this action depends on both the self-dual two-form and the pull-back of the C-field, reducing simply to the volume of Σ when both are zero. Of course, $l \neq 0$ corresponds in the gravity dual to having a non-zero flat C-field. Similarly, in the Type IIA dual picture that we discuss below these are wrapped D4-branes, whose conformal dimensions should be related to the on-shell Dirac-Born-Infeld action, including the B_2-field (3.16). We shall not investigate this further here, but instead leave it for future work.

The remaining four dibaryon operators in (3.23) correspond to the same type of submanifold; hence, without loss of generality, we shall study the $\mathscr{A}_1$ operator. The locus $\{z_1 = \mathrm{i}z_2\}$ in the Calabi-Yau cone X_n cuts out a *singular* subvariety for general n: clearly, z_1 may take any value in $\mathbb{C}$, but the remaining defining equation of X_n implies that $z_0^n + z_3^2 + z_4^2 = 0$, which is a copy of the $\mathcal{A}_{n-1}$ singularity. Thus the divisor of interest is $\mathbb{C} \times (\mathbb{C}^2/\mathbb{Z}_n)$, and the intersection with Y_n is then a copy of the *singular* space $\Sigma_n^{(1)} = S^5/\mathbb{Z}_n$. On the other hand, the $\mathbb{Z}_k$ quotient acts freely on $\Sigma_n^{(1)}$. The volume may again be computed from the character formula [12], giving

$$\mathrm{vol}(\Sigma_n^{(1)}) = \frac{(n+1)^3\pi^3}{8n^2}\,, \tag{3.27}$$

and hence conformal dimension $\Delta[\mathscr{A}_i] = nN/(n+1)$. Again, notice this precisely matches the scaling dimensions of the fields A_i obtained by imposing that the superpotential has scaling dimension 2.

It is instructive to also consider the reduction to Type IIA. The wrapped M5-branes above then become D4-branes wrapped on four-dimensional subspaces $\Sigma_n^{(i)}/\mathrm{U}(1)_b$. Since the quotient by $\mathrm{U}(1)_b$ does not break supersymmetry of the background, we expect that the four-dimensional submanifolds here will also be supersymmetric; however we have not checked the kappa-symmetry of the wrapped D4-branes explicitly.

The reduction of $\Sigma_n^{(0)}$ is diffeomorphic to $S^2 \times S^2$. More interesting is the reduction of the (singular) $\Sigma_n^{(1)}$ subspaces, corresponding to the dibaryonic operators (3.23) with l uncontracted indices. The latter dependence on l may be understood by analysing a certain tadpole in Type IIA, as for the ABJM theory. To discuss the reduction to Type IIA, it is more convenient to use the coordinates A_i, B_i. The divisor corresponding to the $\mathscr{A}_1$ operator is then simply $\{z_1 = \mathrm{i}z_2\} = \{A_1 = 0\}$. The group $\mathrm{U}(1)_b$ acts with charge

-1 on the coordinate B_1, and charges $(1,-1)$ on (A_2, B_2). The $\mathcal{A}_{n-1}$ singularity in these coordinates is $z_0^n + A_2B_2 = 0$. Denoting by u_1, u_2 standard coordinates on $\mathbb{C}^2$ under which $\mathbb{Z}_n$ acts as $(\mathrm{e}^{2\pi i/n}, \mathrm{e}^{-2\pi i/n})$, then the invariant functions under $\mathbb{Z}_n$ are $A_2 = u_1^n$, $B_2 = u_2^n$ and $z_0 = \mathrm{e}^{i\pi/n}u_1u_2$, from which one sees explicitly that $A_2B_2 = -z_0^n$. Thus $\mathrm{U}(1)_b$ acts with weights $(1/n, -1/n)$ on the coordinates (u_1, u_2). This implies that the quotient is topologically $\Sigma_n^{(1)}/\mathrm{U}(1)_b = (S^5/\mathbb{Z}_n)/\mathrm{U}(1)_b \cong \mathbb{WCP}^2_{[n,1,1]}$. The latter is the subspace on which the D4-brane is wrapped. It has an isolated $\mathbb{Z}_n$ orbifold singularity at the image of $A_2 = B_2 = 0$, which lifts to the $\mathcal{A}_{n-1}$ singularity. A simple topological description of $\mathbb{WCP}^2_{[n,1,1]}$ is to take $\mathcal{O}(n) \to \mathbb{CP}^1$, and then collapse the boundary, which is $S^3/\mathbb{Z}_n$, to a point. The latter is then the isolated singularity. Conversely, the image of $B_1 = 0$ is a smooth two-sphere which lifts to the $S^3/\mathbb{Z}_n$ link of the $\mathcal{A}_{n-1}$ singularity. Thus in general the integral of $F_2/(2\pi l_s g_s)$ over this S^2 in $\mathbb{WCP}^2_{[n,1,1]}$ is equal to nk.

Now, from appendix A we have that $H_4(M_n, \mathbb{Z}) \cong \mathbb{Z}$. Call the generator Σ^4. It is also shown in this appendix that the integral of the square of $\Omega_2 = 1 \in H^2(M_n, \mathbb{Z}) \cong \mathbb{Z}$ over Σ^4 is equal to n. Now, in general also $[F_2/2\pi l_s g_s] = k\Omega_2$, and since the first Chern class of $\mathcal{O}(n) \to \mathbb{CP}^1$ is n, it follows that the integral of the pull-back of $\Omega_2 \wedge \Omega_2$ over $\mathbb{WCP}^2_{[n,1,1]}$ is equal to $n^2/n = n$. This implies that the copy of $\mathbb{WCP}^2_{[n,1,1]}$ on which the BPS D4-brane is wrapped is a (singular) representative of the four-cycle Σ^4 in the smooth six-manifold M_n.

Consider now the Wess-Zumino couplings on the D4-brane wrapped on $\mathbb{WCP}^2_{[n,1,1]}$. Due to the presence of the B_2-field (3.16), we obtain[12] the term

$$\frac{1}{(2\pi)^4 l_s^5}\int_{\mathbb{R}_{\mathrm{time}}} A \cdot \int_{\Sigma_4} B_2 \wedge F_2 = l \cdot \frac{g_s}{2\pi l_s^2}\int_{\mathbb{R}_{\mathrm{time}}} A\ . \tag{3.28}$$

Here we have performed the calculation

$$\int_{\Sigma^4} \frac{l}{nk}\Omega_2 \wedge k\Omega_2 = l\ . \tag{3.29}$$

The Wess-Zumino coupling thus induces a tadpole for the worldvolume gauge field A. To cancel this tadpole requires that l fundamental strings end on the D4-brane. In the field theory this corresponds to the fact that the dibaryon operators (3.23) have precisely l uncontracted indices [23].

The alert reader will notice an important subtlety in this argument: in the gravity solution l is defined only modulo nk, while in the field theory $0 \leq l \leq nk$. In particular, when one states that the tadpole requires l fundamental strings to end on the D4-brane, this is only true modulo nk. Thus, it must be that nk fundamental strings are physically equivalent to none. In fact this is easy to see in the M-theory lift. The strings lift to nk M2-branes ending on the M5-brane. More precisely, the end of the M2-branes wrap the M-theory circle that is a smooth S^1 in $\Sigma_n^{(1)}$, together with the time direction in AdS_4. If we remove the singular locus from $\Sigma_n^{(1)}$, which is a copy of S^1, we obtain a smooth

[12]This assumes that the worldvolume gauge field flux on Σ^4 is zero. In fact for *odd* n, the smooth locus of the wrapped submanifold $\Sigma^4 = \mathbb{WCP}^2_{[n,1,1]}$ is not spin, and thus one must turn on a 1/2-integral worldvolume gauge field flux to cancel the resulting Freed-Witten anomaly. This is related to the 1/2-integral shift of B_2 (in the case $n = 1$) in footnote 9, which cancels this. In our case of interest, $n = 2$, there is no such shift.

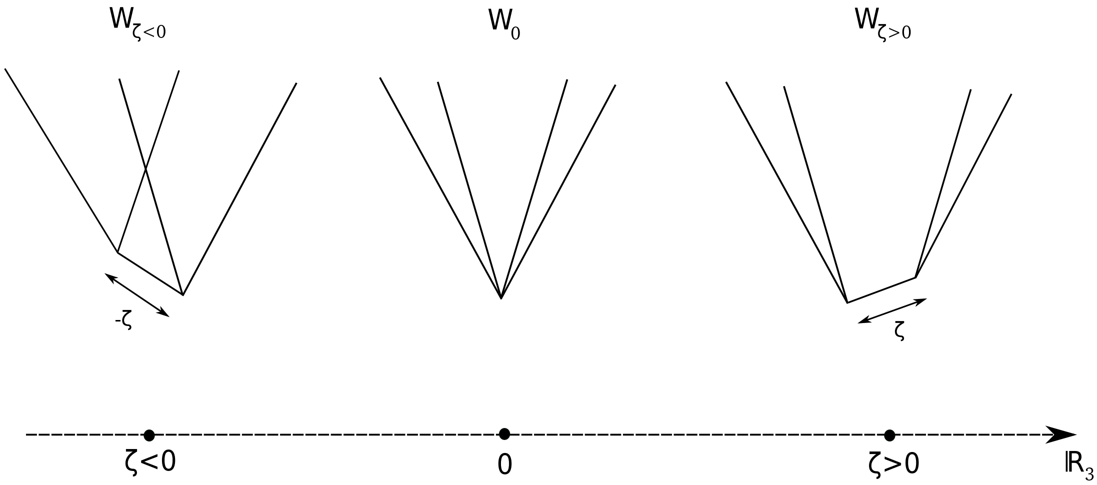

Figure 2. The Type IIA reduction of M-theory on $X/\mathbb{Z}_k$ on $\mathrm{U}(1)_b$ is $C(M_n)$. This geometry may also be viewed as a fibration of W_n^ζ over the $\mathbb{R}_3$ direction, where the size $|\zeta|$ of the exceptional $\mathbb{CP}^1$ depends on the position in $\mathbb{R}_3$. In particular, the conical singularity of $C(M_n)$ is the conical singularity of W_n^0 above the origin in $\mathbb{R}_3$. The above schematic picture would be precisely the toric diagram in the case $n = 1$ (for $n > 1$ the geometry is not toric).

manifold with fundmental group $\mathbb{Z}_{nk}$ — removing the singular locus is sensible, since the supergravity approximation will break down near to this locus. This result implies that nk M2-branes ending on the M5-brane can "slip off", since nk copies of the circle that they wrap are contractible on the M5-brane worldvolume. This matches nicely with the fact that this is equivalent, via (3.28), to a large gauge transformation of the B_2-field.

3.5 Type IIA derivation of the Chern-Simons theories

There is a different way of thinking about the Type IIA backgrounds discussed in section 3.2, which we explain in this section. This demonstrates rather directly the relationship with the "parent" four-dimensional field theories, and elucidates the stringy origin of the Chern-Simons-quiver theories. We will also need the present discussion to derive a Type IIB Hanany-Witten-like brane configuration in the next section.

We begin by considering the geometry $\mathbb{R}^{1,2} \times X_n/\mathbb{Z}_k$ in M-theory, where X_n is the cone singularity (2.9), together with N spacefilling M2-branes. The $\mathrm{U}(1)_b$ circle acts freely away from the cone point, and thus we can reduce to a Type IIA geometry $\mathbb{R}^{1,2} \times C(M_n)$, with k units of RR two-form flux through the generator of $H_2(M_n, \mathbb{Z}) \cong \mathbb{Z}$. In this picture we have N spacefilling D2-branes. However, we may instead take the *Kähler quotient* of $X_n/\mathbb{Z}_k$ by $\mathrm{U}(1)_b$, at level $\zeta \in \mathbb{R}$, to obtain precisely the three-fold W_n^ζ introduced in section 2.4. For $\zeta = 0$, recall this is the affine three-fold given by (2.14), while for $\zeta \neq 0$ one instead obtains Laufer's small resolution of this singularity, which has a blown-up $\mathbb{CP}^1$ of size $|\zeta|$. The latter is the Abelian vacuum moduli space of the four-dimensional parent theory, as discussed in section 2.4. This picture describes the seven-dimensional space $C(M_n)$ as a fibration of W_n^ζ over the real line $\mathbb{R}$ that parametrizes the moment map level ζ, as shown in figure 2.

Indeed, we can instead consider starting with Type IIA on $\mathbb{R}^{1,2} \times \mathbb{R}_3 \times W_n^0$, where we have labelled $\mathbb{R} = \mathbb{R}_3$ for later convenience, with N spacefilling D2-branes. Here W_n^0 should of course be equipped with some kind of Calabi-Yau metric, although we note that from [12] it does not admit a *conical* Calabi-Yau metric for $n > 1$ ($n = 1$ is the conifold). We might imagine W_n^0 as modelling a local singularity in a compact Calabi-Yau manifold, in which case the Calabi-Yau metric here would in any case be incomplete. If we now T-dualize along the (compactified) $\mathbb{R}_3$ direction, then we precisely obtain the Type IIB string theory set-up yielding the four-dimensional parent theory. We may also replace the singular three-fold by its crepant resolution W_n^ζ, thinking of ζ as parametrizing the period of the Kähler form through the exceptional $\mathbb{CP}^1$. We may then turn on k units of RR two-form flux through this $\mathbb{CP}^1$, although in order to preserve supersymmetry it is necessary to also fibre the size of the $\mathbb{CP}^1$ over the $\mathbb{R}_3$ direction — this may be seen by appealing to the reduction of the M-theory solution above. Thus we identify $\mathbb{R}_3 \cong \{\zeta \in \mathbb{R}\}$. If μ_b denotes the moment map for $\mathrm{U}(1)_b$, so that $\mu_b : X_n/\mathbb{Z}_k \to \mathbb{R}_3$, then notice that the inverse image of $\zeta \in \mathbb{R}_3$ is $\mu_b^{-1}(\zeta) = W_n^\zeta$, so that in particular the cone geometry appears at the origin in $\mathbb{R}_3$. By construction, the RR two-form flux may then be identified with the first Chern class $c_1 \in H^2(W_n^\zeta, \mathbb{Z})$ of the $\mathrm{U}(1)_b$ M-theory circle bundle. One can then compute that

$$\frac{1}{2\pi l_s g_s} \int_{\mathbb{C}P^1} F_2 = k \,. \tag{3.30}$$

As explained in [8], the above picture leads to a physical relation between the parent theory and the Chern-Simons theory. If we have N spacefilling D2-branes together with l fractional D4-branes wrapping the (collapsed) $\mathbb{CP}^1$ in W_n^0, the resulting gauge theory is precisely the $\mathcal{A}_1$ quiver theory with superpotential (2.5), with gauge group $\mathrm{U}(N+l)\times\mathrm{U}(N)$ — this is discussed, for example, in [26]. The key result in [8] is that the addition of the k units of RR two-form flux through the $\mathbb{CP}^1$ then induces a Chern-Simons interaction with levels $(k, -k)$ for the two nodes, respectively, via the Wess-Zumino terms on the fractional branes. This leads to a Type IIA string theory *derivation* of our Chern-Simons-quiver theories, starting with the geometric engineering of the parent theory. Also notice that the l fractional D4-branes, wrapped on the collapsed $\mathbb{CP}^1$, will lift to l fractional M5-branes — since the M5-brane is a magnetic source for the G-field, it is thus natural to identify the l units of torsion G-flux with the l fractional M5-branes. Indeed, more precisely, a copy of the exceptional $\mathbb{CP}^1$ at $\zeta > 0$ in figure 2 is the generator of $H_2(M_n, \mathbb{Z}) \cong \mathbb{Z}$, and this lifts to the generator Σ^3 of $H_3(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$, as shown in appendix A. Thus l fractional D4-branes wrapped on the $\mathbb{CP}^1$ lift to l fractional M5-branes wrapped on Σ^3. The latter is then Poincaré dual to l units of torsion G-flux.

4 Type IIB brane configurations

In this section we derive a Hanany-Witten-like brane configuration in Type IIB string theory. This takes the usual form of D3-branes (wrapped on a circle) suspended between 5-branes, except that for $n > 1$ the 5-branes are embedded non-trivially in spacetime; specifically, they are wrapped on holomorphic curves. This will allow us to understand

JHEP12(2009)017

further aspects of the proposed duality, and also derive a field theory duality via a brane creation effect. The reader whose main interest is the deformed $n = 2$ supergravity solution may wish to skip ahead to section 5.

4.1 T-duality to Type IIB: $k = 0$

We begin with the Type IIA background of $\mathbb{R}^{1,2} \times \mathbb{R}_3 \times W_n^\zeta$, with zero RR flux, discussed at the end of the previous section. Here we have included a Kähler class $\zeta \in \mathbb{R}$, which is a free parameter, so that for $\zeta \neq 0$ W_n^ζ is a smooth non-compact Kähler manifold.

For $\zeta = 0$, we are considering the singular three-fold W_n^0. We rewrite the defining equation (2.14) as

$$W_n^0 = \{w_0^{2n} + w_1^2 - uv = 0\} \subset \mathbb{C}^4\,, \tag{4.1}$$

where as before $u = \mathrm{i}w_2 - w_3$, $v = \mathrm{i}w_2 + w_3$. We may then consider performing a T-duality along $\mathrm{U}(1) \equiv \mathrm{U}(1)_6$ that acts with charge 1 on u and charge -1 on v. We may also consider the Kähler quotient by $\mathrm{U}(1)_6$, with moment map $\mu_6 = |u|^2 - |v|^2$, which maps $\mu_6 : W_n^0 \to \mathbb{R} \equiv \mathbb{R}_7$, where we have introduced the subscript 7 to distinguish this copy of $\mathbb{R}$ from $\mathbb{R}_3$ above. It follows that $\{\mathbb{C}^2 = \langle u, v\rangle\}//\mathrm{U}(1)_6 \cong \mathbb{C}$, for any value of μ_6, and hence similarly $W_n^0//\mathrm{U}(1)_6 \cong \mathbb{C}^2$. Indeed, the defining equation of W_n^0 is then $w_0^{2n} + w_1^2 = w$, where $w = uv$ is the coordinate on $\mathbb{C} = \mathbb{C}^2/\mathbb{C}_6^*$. We may thus eliminate the coordinate w to see that $W_n^0//\mathrm{U}(1)_6 \cong \mathbb{C}^2$, spanned by the coordinates w_0, w_1, for any value of the moment map. It follows that $W_n^0/\mathrm{U}(1)_6$ is a $\mathbb{C}^2$ fibration over $\mathbb{R}_7$, and thus $W_n^0/\mathrm{U}(1)_6 \cong \mathbb{R}_7 \times \mathbb{C}^2 \cong \mathbb{R}^5$.

There are, however, fixed points of $\mathrm{U}(1)_6$. If we peform a T-duality along $\mathrm{U}(1)_6$, the above shows that the T-dual spacetime is $\mathbb{R}^{1,2} \times \mathbb{R}_3 \times S_6^1 \times \mathbb{R}_7 \times \mathbb{C}^2$, where S_6^1 is the $\mathrm{U}(1)_6$ circle after performing the T-duality. However, there are codimension four fixed point sets of $\mathrm{U}(1)_6$, where the action on the normal fibre is the standard Hopf action on $\mathbb{R}^4$. These become NS5-branes in the T-dual Type IIB picture. The fixed locus here is $u = v = 0$, which is the origin in the moment map direction $\mathbb{R}_7$. In the $\mathbb{C}^2$ direction they cut out the locus $w_0^{2n} = -w_1^2$ in $\mathbb{C}^2$, which is $w_1 = \pm \mathrm{i}w_0^n$. These are two copies of $\mathbb{C}$ embedded as affine algebraic curves in $\mathbb{C}^2$, which intersect over the origin $\{w_0 = w_1 = 0\}$. Note that when $n = 1$, which is the ABJM case, we see $w_1 = \pm\mathrm{i}w_0$ are two *linearly* embedded copies of $\mathbb{C}$. This is indeed the standard Hanany-Witten brane configuration for the conifold [39]. For $n > 1$, we obtain a *non-linear* version of this, where the NS5-branes are embedded as the curves $w_1 = \pm\mathrm{i}w_0^n$ in $\mathbb{C}^2$. We label the latter directions 4589, and refer to $\mathbb{C}^2_{4589}$. The NS5-branes also sit at a point in the S_6^1 circle, where their distance of separation is the period of the B_2-field through the collapsed $\mathbb{CP}^1$ in W_n^0. The final Type IIB picture is described in figure 3.

Note we can immediately read off the matter content of the field theory from this picture: the brane set-up is identical, apart from the embedding of the NS5-branes in 4589, to the $\mathcal{A}_1$ singularity. Thus we may read off two gauge groups, corresponding to the N D3-branes breaking on the two NS5-branes on the S_6^1 circle. At each NS5-brane we obtain a pair of bifundamentals, A_i, B_i, and an adjoint Φ_1, Φ_2 for each D3-brane

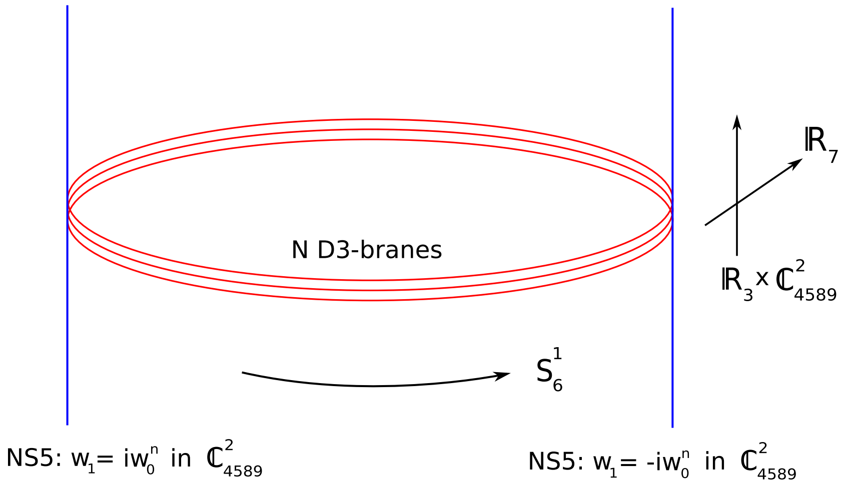

Figure 3. The Type IIB brane dual of the Type IIA background $\mathbb{R}^{1,2}_{012} \times \mathbb{R}_3 \times W^0_n$ with N spacefilling D2-branes. The Type IIB spacetime is flat: $\mathbb{R}^{1,2}_{012} \times \mathbb{R}_3 \times S^1_6 \times \mathbb{R}_7 \times \mathbb{C}^2_{4589}$. There are N D3-branes filling the $\mathbb{R}^{1,2}_{012}$ directions and wrapping the S^1_6 circle; they are at the origin in $\mathbb{R}_3$, $\mathbb{R}_7$ and $\mathbb{C}^2_{4589}$. There are two NS5-branes that are spacefilling in $\mathbb{R}^{1,2}_{012}$ and separated by a distance in the S^1_6 circle that is given by the period of B_2 through the collapsed $\mathbb{CP}^1$ in the T-dual three-fold geometry W^0_n; they both sit at the origin in $\mathbb{R}_7$, fill the $\mathbb{R}_3$ direction, and wrap the holomorphic curves $w_1 = \pm \mathrm{i} w_0^n$, respectively, in $\mathbb{C}^2_{4589}$ with complex coordinates w_0, w_1. These curves intersect at the origin $w_0 = w_1 = 0$. $n = 1$ is the standard Hanany-Witten brane configuration for the conifold singularity, where the NS5-branes are linearly embedded.

segment. The $\mathcal{A}_1$ theory also has the $\mathcal{N} = 4$ cubic superpotential for these fields. For the $\mathcal{A}_1$ theory, both branes are parallel, say at the origin in the 89 plane. For the conifold theory $n = 1$, one brane is in the 45 plane, while the other is in the orthogonal 89 plane. This corresponds to giving a mass to the adjoints, $-\Phi_1^2 + \Phi_2^2$, as shown in [39]. Integrating these out, one obtains the quartic superpotential of Klebanov-Witten. In the general n case, the non-trivial embedding of the NS5-branes in $\mathbb{C}^2_{4589}$ is reflected in the higher order $(-1)^n \Phi_1^{n+1} + \Phi_2^{n+1}$ superpotential term.

4.2 Adding RR-flux/D5-branes: $k \neq 0$

The next step is to turn back on the RR two-form flux, so that $k \neq 0$: this is then the Type IIA dual of M-theory on $X_n/\mathbb{Z}_k$ with N spacefilling M2-branes. As we discussed in section 3.5, supersymmetry also requires that one fibre the parameter ζ over the $\mathbb{R}_3$ direction. Thus, before discussing this, we first consider the effect of turning on the parameter ζ in the T-dual IIB brane set-up above.

Without loss of generality, we take $\zeta > 0$ so that $W^\zeta_n \cong W^+_n$ is biholomorphic to Laufer's resolved manifold, with an exceptional $\mathbb{CP}^1$ replacing the singular point of W^0_n. The $\mathrm{U}(1)_6$ action on W^0_n extends to an action on W^+_n. To see this, recall from the last part of section 2.4 that $(A_1, A_2, B_1, B_2, z_0)$ are coordinates on $\mathbb{C}^5$, and that $x = A_2 B_2$, $y = A_1 B_1$, $u = A_1 B_2$, $v = A_2 B_1$ are invariants under $\mathrm{U}(1)_b$, with $\xi = A_2/A_1$ an invariant on U_1 and $\mu = A_1/A_2$ an invariant on U_2. The embedding equation (2.8) then becomes $x + y + z_0^n = 0$. When $\zeta = 0$ we have the conifold $xy = uv$, and eliminating x this becomes

$y^2 + yz_0^n + uv = 0$, which is the equation $w_1^2 + w_0^{2n} = uv$ of the three-fold W_n^0 on identifying $\mathrm{i}w_1 = y + \frac{1}{2}z_0^n$, $w_0 = 2^{-1/n}z_0$, as before. Thus $\mathrm{U}(1)_6$ rotates u with charge 1 and v with charge -1, and we may lift this to an action on $\mathbb{C}^5$ with coordinates $(A_1, A_2, B_1, B_2, z_0)$ by assigning charges $(1, 0, -1, 0, 0)$. It follows that the charges of (x, y, u, v, ξ, μ) under $\mathrm{U}(1)_6$ are $(0, 0, 1, -1, -1, 1)$. The fixed locus is thus $u = v = \xi = 0$ and $u = v = \mu = 0$ — recall that $\xi = 1/\mu$ on the overlap. Thus on the exceptional $\mathbb{CP}^1$ we fix the north pole $\xi = 0$, and also the south pole $\mu = 0$. We thus see that after resolving W_n^0 to W_n^+ the fixed point set under $\mathrm{U}(1)_6$ is two disjoint copies of $\mathbb{C}$, over the two poles of the $\mathbb{CP}^1$. Indeed, recall that $x = -v\mu - Z_2^n$ on the patch H_2 (where $Z_2 = z_0$), and thus the fixed locus at $v = \mu = 0$ is described by the equation $x = -z_0^n$. Changing variables as above, this becomes precisely $w_1 = -\mathrm{i}w_0^n$. Conversely, the fixed locus $u = \xi = 0$ is the equation $y = -z_0^n$, which under the above change of variable becomes precisely $w_1 = \mathrm{i}w_0^n$.

One can also interpret this in the moment map picture. The moment map is $\mu_6 = |A_1|^2 - |B_1|^2$. Turning on ζ, we also have (2.11). The exceptional $\mathbb{CP}^1$ is, for $\zeta > 0$, at $B_1 = B_2 = 0$. Then the moment map restricted to $\mathbb{CP}^1$ becomes simply $\mu_6 \mid_{\mathbb{CP}^1} = |A_1|^2$. But also $|A_1|^2 = \zeta - |A_2|^2$ on this locus, and thus we see that on $\mathbb{CP}^1$ the moment map ranges from $\mu_6 = 0$ at $A_1 = 0$ to $\mu_6 = \zeta$ at $A_2 = 0$. These are precisely the two poles of the $\mathbb{CP}^1$, which is where the fixed locus is. We thus see that the $\mathbb{CP}^1$ is mapped to an *interval* in the image of the moment map μ_6, which recall is the $\mathbb{R}_7$ direction, with the endpoints of the interval being where the NS5-branes are after performing the T-duality along $\mathrm{U}(1)_6$. Notice that in the holomorphic picture $A_1 = 0$ is the south pole $\mu = 0$ while $A_2 = 0$ is the north pole $\xi = 0$. For negative parameter $\zeta < 0$, the roles of A_i and B_i swap. In this case we will have coordinates $\tilde{\xi} = B_2/B_1$ and $\tilde{\mu} = B_1/B_2$ on the exceptional $\mathbb{CP}^1$, which is now located at $A_1 = A_2 = 0$. The moment map is $\mu_6 \mid_{\widetilde{\mathbb{CP}}^1} = -|B_1|^2$. This ranges from 0 at $B_1 = 0$ to $-\zeta$ at $B_2 = 0$, with the two endpoints being the NS5-brane loci. Notice that the brane at $-\zeta$ is $B_2 = 0$, which is $\tilde{\xi} = 0$, which is the same NS5-brane that moves for $\zeta > 0$, namely that with $w_1 = \mathrm{i}w_0^n$.

To conclude, we see that the T-dual of resolving W_n^0 to W_n^ζ is simply to separate the two NS5-branes in the $\mathbb{R}_7$ direction by a distance ζ — they are wrapped on the same curves as before in the $\mathbb{C}^2_{4589}$ direction. In terms of figure 3, the NS5-brane on the left hand side moves a distance ζ in the (transverse, as drawn) $\mathbb{R}_7$ direction. Notice that once we resolve W_n^0 there is no canonical place to put the D3-branes — we have to pick a point on W_n^ζ. It is natural (in the sense that it preserves a $\mathrm{U}(1) \subset \mathrm{SU}(2)_r$ symmetry) to put them either at the north pole or south pole of the $\mathbb{CP}^1$, in which case the D3-branes intersect either one NS5-brane or the other.

We may now consider what happens when we turn on the RR two-form flux. Recall this fibres the parameter ζ over the $\mathbb{R}_3$ direction in Type IIA. It is simple to see what this does in the IIB brane picture. Consider a fixed point in $\mathbb{R}_3$, which means fixing a particular value for ζ. Then the 5-branes are separated by some distance ζ in the $\mathbb{R}_7$ direction. More precisely, the above analysis shows that for $\zeta > 0$ the 5-brane at the south pole is always at the origin in $\mathbb{R}_7$, while the brane at the north pole is at ζ in $\mathbb{R}_7$. As we move towards the origin in $\mathbb{R}_3$, the 5-branes get closer together in the $\mathbb{R}_7$ direction, until finally at the origin they meet. We may then pass through the origin to $\zeta < 0$, where the behaviour is the

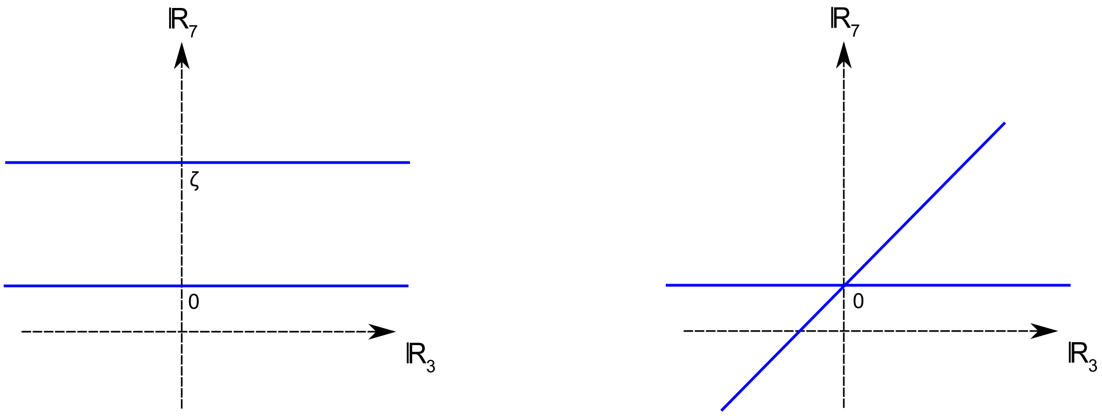

Figure 4. On the left hand side: the positions of the two NS5-branes with resolution parameter ζ in the Type IIA dual. The NS5-brane at position ζ is that wrapped on $w_1 = \mathrm{i}w_0^n$, while the brane at the origin is that wrapped on $w_1 = -\mathrm{i}w_0^n$. On the right hand side: the positions of the 5-branes after turning on the RR flux in the Type IIA dual, which fibres the resolution parameter over the $\mathbb{R}_3$ direction. One of the branes rotates so that they now intersect at the origin of the $\mathbb{R}_3 - \mathbb{R}_7$ plane.

same (with A_i replaced by B_i). This shows that after turning on the RR two-form flux, the 5-branes rotate from being at fixed parallel distance in the $\mathbb{R}_7$ direction (and filling the $\mathbb{R}_3$ direction), to being two lines in the $\mathbb{R}_3 - \mathbb{R}_7$ plane that cross at the origin — see figure 4. This means that, after turning on the RR two-form flux, the 5-branes meet precisely at the origin in $\mathbb{R}^6_{345789}$. although they are still non-trivially holomorphically embedded in $\mathbb{C}^2_{4589}$ as $w_1 = \pm\mathrm{i}w_0^n$.

Notice that for $n = 1$ the above indeed reproduces the Type IIB brane picture in ABJM [4] — up to two important details. First, in the case $n = 1$ we have derived the Type IIB brane dual by starting with $\mathbb{C}^4/\mathbb{Z}_k$, reducing to Type IIA along $\mathrm{U}(1)_b$ and then T-dualizing to Type IIB along $\mathrm{U}(1)_6$. In [4], the authors instead began with the Type IIB brane picture, and argued that T-dualizing to Type IIA and uplifting to M-theory gave a non-trivial hyperkähler eight-manifold as the uplift, which is characterized by two harmonic functions, defined on two copies of $\mathbb{R}^3$. The difference between these two pictures is that the former is simply the *near-brane* limit of the latter. Indeed, ABJM showed explicitly that the near-horizon limit of the hyperkähler manifold indeed gives $\mathbb{C}^4/\mathbb{Z}_k$, which amounts to dropping the non-zero constant term in the harmonic functions. This is the dual geometry in the region near to where the 5-branes intersect at the origin in $\mathbb{R}^6_{345789}$ (which are the two copies of $\mathbb{R}^3$ mentioned above).

Second, and more importantly, in the ABJM brane picture the rotated 5-brane in figure 4 is in fact a *bound state* of an NS5-brane with k D5-branes — the latter is effectively the T-dual of the k units (3.30) of RR two-form flux through the (fibred) exceptional $\mathbb{CP}^1$ in the Type IIA geometry. To see the presence of the k D5-branes in the $(1, k)$5-brane bound state directly is not straightforward in the discussion we have given above. However, the k units of D5-brane charge can be seen indirectly by considering a certain tadpole. Thus, we begin in Type IIA on $C(M_6)$, which recall may also be thought of as W_n^ζ fibred over

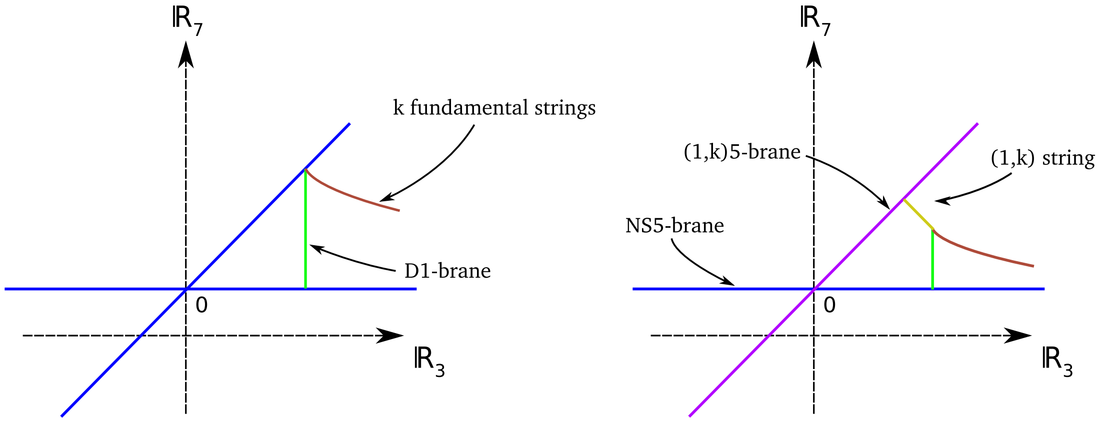

Figure 5. On the left hand side: the naive T-dual configuration to a D2-brane wrapped on the $\mathbb{CP}^1$ at a fixed non-zero point in $\mathbb{R}_3$ is a D1-brane stretching between the two NS5-branes, with k fundamental strings also ending on the D1-brane and one of the NS5-branes to cancel the tadpole. On the right hand side: the correct T-dual configuration, in which the D1-brane and k fundamental strings form a $(1,k)$ string bound state, which then must necessarily end on a $(1,k)$5-brane. (Notice that the D1-brane must also wind around the S^1_6 circle as one moves from one 5-brane to the other along its worldvolume.)

$\mathbb{R}_3$. Pick a non-zero point in $\mathbb{R}_3$, and consider the exceptional $\mathbb{CP}^1$ of size $|\zeta|$ in W_n^ζ over this point. If we wrap a D2-brane over this $\mathbb{CP}^1$, we get a point particle in $\mathbb{R}^{1,2}_{012}$. However, because of the k units of RR two-form flux (3.30) through this $\mathbb{CP}^1$, in fact this configuration does not exist in isolation: one must have k fundamental strings *ending* on the wrapped D2-brane. To see this, note the Wess-Zumino coupling on the D2-brane:

$$\frac{1}{(2\pi)^2 l_s^3}\int_{\mathbb{R}_{\rm time}} A \int_{\mathbb{CP}^1} F_2 = k\cdot\frac{g_s}{2\pi l_s^2}\int_{\mathbb{R}_{\rm time}} A\ . \tag{4.2}$$

To cancel this tadpole, we precisely require k fundamental strings to end at a point on the $\mathbb{CP}^1$.

Consider the T-dual to this in Type IIB. As already discussed, the exceptional $\mathbb{CP}^1$ maps to an interval in the $\mathbb{R}_7$ direction, between the two 5-branes: this lies at the chosen point in $\mathbb{R}_3$, and is at the origin in $\mathbb{C}^2_{4589}$. A D2-brane wrapped on the $\mathbb{CP}^1$ thus T-dualizes to a D1-brane stretched between the two 5-branes in the $\mathbb{R}_7$ direction. The k fundamental strings ending on the D2-brane T-dualize to k fundamental strings ending on the D1-brane. In particular, the fundamental strings may end at one of the poles of the $\mathbb{CP}^1$. In the IIB picture, we therefore have a D1-brane and also k fundamental strings terminating on *one* of the 5-branes (while for the other 5-brane there is only a D1-brane ending on it). In general, a (p,q) string, where p denotes the number of D1-branes and q the number of fundamental strings in a bound state string, can only end on a (p,q)5-brane. Thus the only way to make sense of the above tadpole is that the 5-brane is in fact a $(1,k)$5-brane, and the D1-brane and k fundamental strings form a $(1,k)$ bound state ending on this. Of course, this precisely reproduces the correct brane configuration of ABJM in the case of $n=1$.

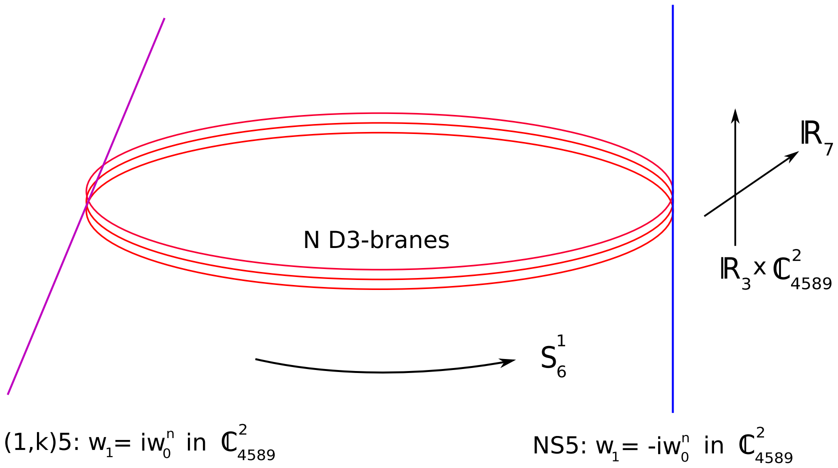

Figure 6. The final Type IIB dual of M-theory on $X_n/\mathbb{Z}_k$. The spacetime is $\mathbb{R}^{1,2}_{012} \times \mathbb{R}_3 \times S^1_6 \times \mathbb{R}_7 \times \mathbb{C}^2_{4589}$. There are N D3-branes filling the $\mathbb{R}^{1,2}_{012}$ directions and wrapping the S^1_6 circle; they are at the origin in $\mathbb{R}_3$, $\mathbb{R}_7$ and $\mathbb{C}^2_{4589}$. There are also two spacefilling 5-branes in $\mathbb{R}^{1,2}_{012}$ at points on the S^1_6 circle. The first is an NS5-brane, sitting at the origin in $\mathbb{R}_7$ and filling $\mathbb{R}_3$, which wraps the curve $w_1 = -\mathrm{i}w_0^n$ in $\mathbb{C}^2_{4589}$. The second is a $(1,k)$5-brane, wrapping an angled line through the origin in the $\mathbb{R}_3 - \mathbb{R}_7$ plane, and wrapping the curve $w_1 = \mathrm{i}w_0^n$ in $\mathbb{C}^2_{4589}$.

To conclude, we have shown that M-theory on $X_n/\mathbb{Z}_k$ has a Type IIB dual of Hanany-Witten type: it is identical to the brane set-up for $n = 1$ described by ABJM [4], except that the 5-branes are wrapped on the holomorphic curves $w_1 = \pm \mathrm{i}w_0^n$ inside $\mathbb{C}^4_{4589}$ — see figure 6.

4.3 Brane creation effect

Having described the Type IIB brane dual, an important dynamical question is what happens when we move the two 5-branes past each other on the S^1_6 circle. This was first studied by Hanany-Witten [14], and the analysis in section 5 of that paper may be applied directly to the case $n = 1$ (the ABJM case). We thus begin by describing the $n = 1$ case, and then explain how to apply this result for $n > 1$ by deforming the curves in $\mathbb{C}^2_{4589}$ so that the brane intersections in $\mathbb{R}^6_{345789}$ are normal crossings.

We thus start with $n = 1$. We suppress the spacetime $\mathbb{R}^{1,2}_{012}$ from the discussion, since all branes are spacefilling in these directions. Thus the relevant geometry is $S^1_6 \times \mathbb{R}^6_{345789}$. We have an NS5-brane at a point $0 \neq t \in S^1_6$ and at the origin in 789, and a $(1,k)$5-brane at the origin $0 \in S^1_6$ and at the origin in 345. Notice that we have, for convenience of notation, rotated the axes relative to figure 6: the argument we are about to give is entirely topological, and so is unaffected. We denote these submanifolds as $W_{\mathrm{NS,t}}$ and $W_{(1,k)}$, respectively. These two copies of $\mathbb{R}^3$ that are wrapped by the 5-branes thus intersect normally at the origin in $\mathbb{R}^6_{345789}$. However, importantly, the branes do not actually intersect in spacetime unless $t = 0$.

The $(1,k)$5-brane sources k units of RR three-form flux F_3 through a sphere S^3 linking its worldvolume. Thus, let S^3 be a normal sphere around a point on the $(1,k)$-brane in

$S^1_6 \times \mathbb{R}^6_{34578}$, so that

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{S^3} F_3 = k \ . \tag{4.3}$$

Following [14], we then define the *linking number*

$$L_t = \frac{1}{(2\pi l_s)^2 g_s} \int_{W_{\mathrm{NS,t}}} F_3 \ . \tag{4.4}$$

This is independent of t as t is varied, *provided* we do not cross the origin $t = 0$. The reason for this is that F_3 is closed on the complement of the $(1,k)$5-brane worldvolume, and the independence of (4.4) on t then follows from Stokes' Theorem. More precisely, $\mathrm{d}F_3$ is a four-form which is supported only on the $(1,k)$5-brane worldvolume at $t = 0$ and the origin in 345: it is k times a delta-function representative of the Poincaré dual of $W_{(1,k)}$.

Consider now moving the NS5-brane from $t_+ > 0$, on the right of the $(1,k)$5-brane, to $t_- < 0$ on the left. Let $I = [t_-, t_+]$ be the interval in the S^1_6 circle covered in this motion. Then we have linking numbers (4.4) L_+ and L_- on the right and left. We may compute the change in linking number using Stokes' Theorem:

$$L_+ - L_- = \frac{1}{(2\pi l_s)^2 g_s} \int_{W_{\mathrm{NS}} \times I} \mathrm{d}F_3 = k \ . \tag{4.5}$$

On the worldvolume of the NS5-brane there is a U(1) gauge field A_{NS}, with field strength F_{NS}, and it is only the combination $\Lambda = C_2 - 2\pi l_s^2 F_{\mathrm{NS}}$ that is gauge invariant. Moreover,

$$F_3 \mid_{W_{\mathrm{NS}}} = \mathrm{d}\Lambda \, , \tag{4.6}$$

meaning that F_3 must be exact on the NS5-brane worldvolume $W_{\mathrm{NS,t}}$. In the non-compact setting of interest, of course all closed forms are exact on $W_{\mathrm{NS,t}} \cong \mathbb{R}^3$, so (4.6) is always satisfied. However, what we learn from (4.5) is that the period of F_3 through $W_{\mathrm{NS,t}}$ changes by k units as we move the NS5-brane from the right $t > 0$ to the left $t < 0$ of the $(1,k)$5-brane. The explanation for this is that k spacefilling D3-branes are created at the intersection point $t = 0$ when the branes are moved past each other. Indeed, such a D3-brane ending on the NS5-brane is a delta-function source for F_{NS}:

$$\frac{1}{2\pi g_s} \mathrm{d}F_{\mathrm{NS}} = \pm \delta(p) \tag{4.7}$$

where $p \in W_{\mathrm{NS}} \cong \mathbb{R}^3$. That is, the D3-brane ending on the NS5-brane is a magnetic monopole for this U(1) gauge field. The sign in (4.7) depends on whether the D3-brane ends from the right or from the left on the S^1_6 circle, which it wraps (a monopole or anti-monopole). Integrating k times (4.7) over W_{NS} precisely accounts for the change in linking number (4.5). This is the Hanany-Witten effect.

Having carefully reviewed this effect, we may now apply it to the case with $n > 1$. However, note that for $n > 1$ the branes are not linearly embedded in $\mathbb{C}^2_{4589}$: they cross at a single point at the origin, but they are wrapped on non-trivial curves. We may remedy this by *deforming* the curves that the 5-branes are wrapped on. Thus, we change

$$w_1 = -\mathrm{i}w_0^n \longrightarrow w_1 = -\mathrm{i}\prod_{i=1}^{n}(w_0 - \alpha_a) + \alpha_0 \tag{4.8}$$

$$w_1 = \mathrm{i}w_0^n \longrightarrow w_1 = \mathrm{i}\prod_{i=1}^{n}(w_0 - \beta_a) + \beta_0 \ . \tag{4.9}$$

Here α_a, β_a, $a = 0, \ldots, n$, are arbitrary parameters. The point of these deformations is that (a) they preserve the boundary conditions at infinity, since we have added only lower order terms to the polynomials, and (b) the resulting curves now intersect normally in $\mathbb{C}^2_{4589}$. Indeed, these two curves in $\mathbb{C}^2_{4589}$ intersect where the w_1 coordinate in (4.8) equals the w_1 coordinate in (4.9). This results in the nth order polynomial

$$\mathrm{i}\prod_{i=1}^{n}(w_0 - \alpha_a) + \mathrm{i}\prod_{i=1}^{n}(w_0 - \beta_a) - \alpha_0 + \beta_0 = 0 \ . \tag{4.10}$$

For *generic* values of the parameters α_a, β_b, this will have precisely n solutions for w_0, say $w_0^{(i)}$, $i = 1, \ldots, n$. Thus the resulting curves generically intersect at n points $(w_0^{(i)}, w_1^{(i)})$, where of course $w_1^{(i)}$ is given by (4.8) (or (4.9)) evaluated at $w_0^{(i)}$. Moreover, the intersects of the curves near to these n points look precisely like the linear $n = 1$ case.

We are now in good shape: after this generic deformation that preserves the boundary conditions of the branes at infinity, the two branes intersect ordinarily at n points in $\mathbb{R}^6_{345789}$ (they always cross at the origin of the $\mathbb{R}_3 - \mathbb{R}_7$ plane). The above discussion of the Hanany-Witten effect shows that the creation of the k D3-branes as an NS5-brane crosses a $(1, k)$5-brane occurs entirely *locally* at the points where the branes intersect in spacetime. Thus if we move our deformed NS5-brane past the deformed $(1, k)$5-brane, we obtain precisely n copies of the $n = 1$ result, i.e. in total nk D3-branes are created as they are moved past each other. More precisely, k D3-branes are created at each of the n points $(w_0^{(i)}, w_1^{(i)})$ (at the origin in the $\mathbb{R}_3 - \mathbb{R}_7$ plane, and stretched along the S^1_6 circle). Notice that this result is independent of the choice of deformation parameters α_a, β_a, as it is topological. Thus after moving the branes past each other we may deform back to $\alpha_a = \beta_a = 0$, where the nk created D3-branes are all at the origin in $\mathbb{R}^6_{345789}$.

4.4 The field theory duality

The brane creation effect described in the last section leads to an interesting field theory duality, discussed for the ABJM theory in [23, 32]. Here we briefly describe the situation for general n. We begin with the Type IIB brane set-up corresponding to the gauge group $\mathrm{U}(N + l)_k \times \mathrm{U}(N)_{-k}$. This is shown on the left hand side of figure 7.

Consider, without loss of generality, moving the NS5-brane around the circle. Rotating it anti-clockwise by one revolution, as shown on the right hand side of figure 7, the gauge groups become $\mathrm{U}(N)_k \times \mathrm{U}(N + nk - l)_{-k}$. In particular, we note that the $\mathrm{U}(N + nk)_k \times \mathrm{U}(N)_{-k}$ theory can be deformed to the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ theory in this way, which is the required field theory duality to match the dual supergravity analysis mentioned at the very

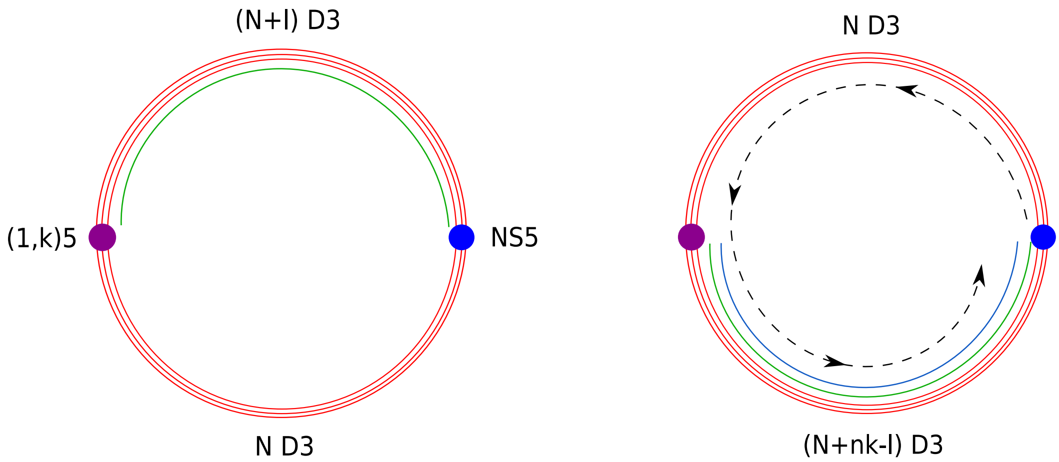

Figure 7. On the left hand side: the initial brane configuration, with $(N+l)$ D3-branes suspended between the 5-branes on one side of the S^1_6 circle, and N D3-branes on the other. On the right hand side: moving the NS5-brane anti-clockwise around the circle pulls the l fractional branes with it. After passing the $(1,k)$5-brane these swap orientation, becoming l anti-branes, and in addition nk D3-branes are created.

end of section 2.2. Moving the NS5-brane multiple times around the circle, or in the other direction, apparently leads to further equivalences, as observed for the $n=1$ ABJM theory in [23]. This certainly deserves further careful study of the brane system to understand properly, although we shall make some comments on this in section 6.2.

5 The deformed supergravity solution

In this section we describe a supergravity solution [15] which is a deformation of the $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ M-theory background discussed in section 3.1, in the sense that it approaches the latter asymptotically at infinity. Throughout this section we set $n=2$. We also begin with $k=1$, and restore general k later.

5.1 The Stenzel metric on T^*S^4

We begin by describing a deformation of the Calabi-Yau cone metric on the quadric cone X_2. The latter has an isolated singularity at $z_0 = \cdots = z_4 = 0$ that may be *deformed*[13] to a smooth non-compact Calabi-Yau variety $\mathcal{X}$, diffeomorphic to T^*S^4 (the cotangent bundle of S^4), via

$$\mathcal{X} \equiv \left\{ \sum_{i=0}^{4} z_i^2 = \gamma^2 \right\} , \tag{5.1}$$

where $\gamma \in \mathbb{C}$ is a constant. For $\gamma \neq 0$ this describes a smooth complex structure on T^*S^4. The deformation breaks the $\mathbb{C}^* \cong \mathbb{R}_+ \times \mathrm{U}(1)_R$ symmetry of the cone to $\mathbb{Z}_2 \subset \mathrm{U}(1)_R$. Using

[13]In the same sense as the more familiar deformed conifold in six dimensions.

the broken $\mathrm{U}(1)_R$ action we take $\gamma \in \mathbb{R}_+$ in what follows. The $S^4 = \mathrm{SO}(5)/\mathrm{SO}(4)$ zero-section is then realized as the real locus of $\mathcal{X}$ in $\mathbb{C}^5$. The cotangent bundle structure may be seen explicitly by writing

$$z_i = \cosh\left(\sqrt{p_j p_j}\right) x_i + \frac{\mathrm{i}}{\sqrt{p_j p_j}} \sinh\left(\sqrt{p_j p_j}\right) p_i \ . \tag{5.2}$$

Then $\sum_{i=0}^4 x_i^2 = \gamma^2$, $\sum_{i=0}^4 x_i p_i = 0$, so that the S^4 is $\{p_i = 0\}$.

There is an explicit complete Ricci-flat Kähler metric on $\mathcal{X}$ which is asymptotic to the cone metric at large radius, called the Stenzel metric. This is cohomogeneity one under the action of SO(5), with principal orbits diffeomorphic to $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$, and degenerate special orbit $S^4 = \mathrm{SO}(5)/\mathrm{SO}(4)$. The Kähler structure induces the standard symplectic structure on T^*S^4, and thus the S^4 is Lagrangian; in fact it is special Lagrangian, and is thus a minimal volume representative of the generator of $H_4(\mathcal{X}, \mathbb{Z}) \cong \mathbb{Z}$. Note that given any Ricci-flat metric $\mathrm{d}s^2$, the rescaled metric $\gamma^2 \mathrm{d}s^2$ is also Ricci-flat, for any positive constant $\gamma \in \mathbb{R}_+$, and this is essentially the constant γ above, which is proportional to the radius of the S^4.

In terms of invariant one-forms on the coset space $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$, the metric on $\mathcal{X}$ may be written as

$$\mathrm{d}s^2_{\mathcal{X}} \ = c^2 \mathrm{d}r^2 + c^2\nu^2 + a^2 \sum_{i=1}^3 \sigma_i^2 + b^2 \sum_{i=1}^3 \tilde{\sigma}_i^2 \, , \tag{5.3}$$

where

$$\begin{aligned} a^2 &= \frac{1}{3}(2+\cosh 2r)^{1/4} \cosh r \, , \qquad b^2 = \frac{1}{3}(2+\cosh 2r)^{1/4} \sinh r \tanh r \, , \\ c^2 &= (2+\cosh 2r)^{-3/4} \cosh^3 r \ . \end{aligned} \tag{5.4}$$

More details may be found in appendix B. In these coordinates, the S^4 is located at $r = 0$. Note here we have picked a particular representative metric in the conformal class of metrics on $\mathcal{X}$, i.e. a particular value of γ. It will be straightforward to reintroduce this scale later. The calibrated S^4 in the above solution has fixed size, with induced round metric

$$\mathrm{d}s^2_{S^4} = 3^{-3/4}(\nu^2 + \sum_{i=1}^3 \sigma_i^2) \ . \tag{5.5}$$

After a change of variable

$$\rho^2 \sim \frac{16}{9}\frac{1}{2^{9/4}} \mathrm{e}^{\frac{3}{2}r} \, , \tag{5.6}$$

the asymptotic form of the metric is

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \rho^2 \left[\frac{3}{8} \sum_{i=1}^3 \left(\sigma_i^2 + \tilde{\sigma}_i^2\right) + \frac{9}{16}\nu^2 + \frac{2^{1/3}}{3^3} \frac{1}{\rho^{8/3}} \sum_{i=1}^3 \left(\sigma_i^2 - \tilde{\sigma}_i^2\right) + \ldots \right] \ . \tag{5.7}$$

The leading term is the metric on the cone over the manifold $Y_2 = V_{5,2}$.

For later use we record here the results of certain integrals. Noticing that the S^4 is parametrized by ν, σ_i, and recalling that $V_{5,2}$ is an S^3 bundle over S^4, we have

$$\int_{S^3_{\text{fibre}}} \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 = 2\pi^2 \ . \tag{5.8}$$

This is the volume of a unit S^3, as necessarily follows since the collapse of this S^3 at the S^4 zero-section is regular. Writing the volume form of $V_{5,2}$ as

$$\mathrm{dvol}_{V_{5,2}} \ = \ \frac{3^4}{2^{11}} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \wedge \nu \, , \tag{5.9}$$

and using the total volume of $V_{5,2}$ (3.5), we deduce also that

$$\int_{S^4} \nu \wedge \sigma_1 \wedge \sigma_1 \wedge \sigma_3 = \frac{8\pi^2}{3} \, , \tag{5.10}$$

which is in fact the volume of a unit radius round S^4.

5.2 The deformed M2-brane solution

The $\mathrm{AdS}_4 \times V_{5,2}$ supergravity solution admits a smooth supersymmetric deformation, based on the above Stenzel metric. This solution was presented in [15]. We have found and corrected a few minor mistakes in the formulas in [15], which are important for the physical interpretation. The $d = 11$ solution is[14]

$$\begin{aligned} \mathrm{d}s^2 \ &= \ H^{-2/3} \mathrm{d}s^2_{\mathbb{R}^{1,2}} + H^{1/3} \gamma^2 \mathrm{d}s^2_{\mathcal{X}} \, , \\ G \ &= \ \mathrm{d}^3 x \wedge \mathrm{d}H^{-1} + m\alpha \, , \end{aligned} \tag{5.11}$$

where m is a constant, $\mathrm{d}s^2_{\mathcal{X}}$ denotes the Stenzel metric, and α is a harmonic *self-dual* four-form on $\mathcal{X}$ [15]. In terms of the orthonormal frame (B.4) defined in appendix B this reads

$$\alpha \ = \ \frac{3}{\cosh^4 r} \left(e^{\tilde{0}123} + e^{0\tilde{1}\tilde{2}\tilde{3}} \right) + \frac{1}{2} \frac{1}{\cosh^4 r} \epsilon_{ijk} \left(e^{0ij\tilde{k}} + e^{\tilde{0}\tilde{i}\tilde{j}\tilde{k}} \right) \ . \tag{5.12}$$

More precisely, this is an L^2-normalizable primitive harmonic $(2,2)$-form on $\mathcal{X}$. Note that α generates $H^4_{\mathrm{cpt}}(\mathcal{X}, \mathbb{R}) \cong \mathbb{R}$. By the general results of [40], this is the only L^2-normalizable harmonic form on $\mathcal{X}$ in fact. The equation of motion for the G-field

$$\mathrm{d} * G \ = \ \frac{1}{2} G \wedge G \, , \tag{5.13}$$

implies the following equation for the warp factor

$$\Delta_{\mathcal{X}} H \ = \ -\frac{12 m^2}{\cosh^8 r} \ . \tag{5.14}$$

[14] We have introduced an explicit deformation parameter γ which is set to unity in [15]. This measures the radius of the S^4 at the origin.

Here $\Delta_{\mathcal{X}}$ denotes the scalar Laplacian on the Stenzel manifold with metric $\mathrm{d}s^2_{\mathcal{X}}$. This can be integrated explicitly in terms of the variable $y^4 = 2 + \cosh 2r$, giving

$$H(y) = \frac{-24m^2}{\sqrt{2}} \int \frac{\mathrm{d}y}{(y^4-1)^{5/2}}\,, \tag{5.15}$$

where an integration constant has been fixed by requiring regularity near to $r = 0$. In terms of the variable ρ introduced in (5.6), the asymptotic expansion reads

$$H(\rho) = \frac{2^{10}}{3^5}\frac{m^2}{\rho^6} + \ldots \qquad \text{for } \rho \to \infty\ . \tag{5.16}$$

Notice that this has a different behaviour from the Klebanov-Strassler solution, where one has logarithmic corrections. As explained in [15], this difference comes from the fact that the self-dual harmonic form is *normalizable* here, while it is not normalizable in six dimensions. At large ρ the solution becomes of the form (3.6), where here the AdS$_4$ radius is expressed in terms of the integration constant m^2 as $R^6 = \frac{2^{10}}{3^7}m^2$.

5.3 The G-flux

We now wish to discuss the quantization of the flux, thus relating the constant m^2 to the quantized fluxes. Because the background is asymptotically AdS$_4 \times V_{5,2}$, it is natural to quantize the flux of $*G$ through the $V_{5,2}$ at infinity, as in (3.7), and interpret this as the number of M2-branes in the UV. More generally, we may define a "running" number of M2-branes $N(r)$ as

$$N(r) = \frac{1}{(2\pi l_p)^6}\int_{Y_r} *G\,, \tag{5.17}$$

where the integral is evaluated on a seven-dimensional surface of constant r, which is a copy of $V_{5,2}$. To compute this, we may use the four-form equation of motion (5.13) to write

$$\int_{Y_2^r} *G = \frac{1}{2}\int_{\mathcal{X}^r} G\wedge G = \frac{1}{2}\int_{\mathcal{X}^r} m^2|\alpha|^2 \mathrm{dvol}_{\mathcal{X}}\,, \tag{5.18}$$

where the integral is evaluated on the Calabi-Yau $\mathcal{X}$ cut off at a distance r. The result is

$$N(r) = \frac{1}{(2\pi l_p)^6}\frac{m^2}{9}\frac{2^{11}}{3^4}\mathrm{vol}(V_{5,2})\tanh^4 r\ . \tag{5.19}$$

We see that this running number of M2-branes becomes a constant at infinity, where

$$N \equiv N(\infty) = \frac{1}{(2\pi l_p)^6}\frac{2^{11}}{3^6}m^2\mathrm{vol}(V_{5,2})\ . \tag{5.20}$$

This determines m^2 in terms of the physical paramater N. Eliminating m^2 we see that the (UV) AdS$_4$ radius takes exactly the form (3.8).

We are not quite done, however. There is a non-trivial cycle in the geometry, namely the four-sphere at the zero-section of $\mathcal{X} = T^*S^4$. Thus we have to impose the quantization of the four-form flux through this cycle. Noting that the restriction of the $(2,2)$ four-form α to a four-sphere at any distance r from the origin is

$$\alpha|_{S^4_r} = \frac{1}{\sqrt{3}\cosh r}\nu \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3\,, \tag{5.21}$$

we compute

$$\frac{1}{(2\pi l_p)^3}\int_{S^4} G \;=\; \frac{1}{(2\pi l_p)^3}\frac{m}{\sqrt{3}}\frac{8\pi^2}{3} \;=\; \tilde{M} \in \mathbb{N}\,, \tag{5.22}$$

where recall that the volume of the unit S^4 at the origin is $8\pi^2/3$. The reason for denoting the integer[15] flux as $\tilde{M}$ will become clear momentarily. We hence obtain another expression for m^2, namely $m^2 = 27\pi^2 l_p^6 \tilde{M}^2$. The running number of M2-branes then takes the simple form

$$N(r) \;=\; \frac{\tilde{M}^2}{4}\tanh^4 r\;. \tag{5.23}$$

There is a simple way to check the numerical factor here. If we integrate (5.13) over the whole of $\mathcal{X}$, the left hand side gives $(2\pi l_p)^6 N$. On the other hand, the right hand side is a topological quantity. To see this, note that the integral of G over S^4 is by definition $(2\pi l_p)^3\tilde{M}$. But we may also regard G as defining an element of $H^4_{\mathrm{cpt}}(\mathcal{X},\mathbb{R})$. The map $\mathbb{R} \cong H^4_{\mathrm{cpt}}(\mathcal{X},\mathbb{R}) \to H^4(\mathcal{X},\mathbb{R}) \cong \mathbb{R}$ is just multiplication by 2, the latter being the Euler number of S^4. Then we may interpret $\frac{1}{2}\int_{\mathcal{X}} G\wedge G$ as the cup product $H^4(\mathcal{X},\mathbb{R}) \times H^4_{\mathrm{cpt}}(\mathcal{X},\mathbb{R}) \to H^8_{\mathrm{cpt}}(\mathcal{X},\mathbb{R}) = \mathbb{R}$ via $\frac{1}{2}[G]\cup[G]_{\mathrm{cpt}} = (2\pi l_p)^6\frac{1}{2}\tilde{M}\cdot\frac{\tilde{M}}{2}$. This is a simple topological check on (5.23).

Since we have $N = \tilde{M}^2/4$, and N must be an integer, we have to set $\tilde{M} = 2M$. We thus obtain the relation

$$N \;=\; M^2\,, \tag{5.24}$$

where $2M$ is the number of units of G-flux through the S^4 (5.22). Notice that the higher derivative X_8 term in M-theory would lead to a $O(1/N)$ correction to this formula. In fact an explicit solution, generalizing that above and including the X_8 correction, was given in [41].[16] Of course, the supergravity solution is only valid at large N (and hence large M) in any case, and this term is a subleading correction.

As a consequence of the relation $\tilde{M} = 2M$ we also see that there is *no* torsion G-flux turned on in $H^4(V_{5,2},\mathbb{Z}) \cong \mathbb{Z}_2$. To see this we recall that there is a relation between the cohomology of the deformed space $\mathcal{X}$ and the cohomology of its boundary $\partial\mathcal{X} = V_{5,2}$. The only non-trivial cohomology of $\mathcal{X}$ is $H^4(\mathcal{X},\mathbb{Z}) \cong H_4(\mathcal{X},\mathbb{Z}) \cong \mathbb{Z}$, the latter being generated by the S^4 zero-section. There is a map $\mathbb{Z} \cong H^4(\mathcal{X},\mathbb{Z}) \to H^4(V_{5,2},\mathbb{Z}) \cong \mathbb{Z}_2$ induced by restriction to $V_{5,2} = \partial\mathcal{X}$ which is simply reduction modulo 2. The calculation (5.22) means

[15] It is again important here that the membrane anomaly on $\mathcal{X}$ vanishes. This follows from the fact that $w_4(\mathcal{X})\mid_{S^4}$ is twice the fourth Stiefel-Whitney class of the bundle TS^4, and hence zero mod 2 (the latter Stiefel-Whitney class also happens to be zero).

[16] Although some errors in [15] have propagated to this reference.

that as a cohomology class $[G] = 2Me$, where e denotes the generator of $H^4(\mathcal{X}, \mathbb{Z})$. This then maps $[G] \to 0 \in H^4(V_{5,2}, \mathbb{Z}) \cong H_3(V_{5,2}, \mathbb{Z}) \cong \mathbb{Z}_2$.

We may also define a "running C-field period". Recall that $V_{5,2}$ may be thought of as an S^3 bundle over S^4. Then the generator of $H_3(V_{5,2}, \mathbb{Z}) \cong H^4(V_{5,2}, \mathbb{Z}) \cong \mathbb{Z}_2$ may be taken to be a copy of the S^3 fibre at a fixed point on the base S^4. We can identify the torsion three-cycle at a distance r as the three-sphere at a distance r from the origin of the fibre $\mathbb{R}^4$, at a fixed point on S^4. We have

$$\alpha|_{\mathbb{R}^4} = \frac{\sinh^3 r}{\sqrt{3}\cosh^4 r} \mathrm{d}r \wedge \tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 \,, \tag{5.25}$$

and thus

$$c_3(r) \equiv \frac{1}{(2\pi l_p)^3} \int_{S^3_r} C = \frac{m}{(2\pi l_p)^3} \int_{\mathbb{R}^4_r} \alpha = \frac{M}{2} \left[\frac{1}{\cosh r} \left(\frac{1}{\cosh^2 r} - 3 \right) + 2 \right] . \tag{5.26}$$

Notice that $c_3(\infty) = M$. Indeed, this is again purely a topological integral, namely $(1/(2\pi l_p)^3) \int_{\mathbb{R}^4_{\mathrm{fibre}}} G = M$, and shows that the holonomy of the C-field on $V_{5,2}$ at infinity is indeed trivial, *cf* (3.9).

5.4 The $\mathbb{Z}_k$ quotient

If we wish to consider deformations of the $V_{5,2}/\mathbb{Z}_k$ supergravity background with $k > 1$, the deformed solution $\mathcal{X}/\mathbb{Z}_k$ is then singular, having two isolated $\mathbb{C}^4/\mathbb{Z}_k$ singularities at the north p_N and south p_S poles of the S^4 zero-section. Since we cannot trust the supergravity solution near to these points, we should remove them from the spacetime in any supergravity analaysis. It then makes sense to analyse flux quantization on the smooth manifold $(\mathcal{X}\backslash\{p_N, p_S\})/\mathbb{Z}_k$. This has a boundary with three connected components: $V_{5,2}/\mathbb{Z}_k$ at infinity, and two copies of $S^7/\mathbb{Z}_k$ near to $r = 0$.

Since $H_4(\mathcal{X}, \mathbb{Z}) \cong \mathbb{Z}$, generated by the S^4 zero-section, it follows from a simple Mayer-Vietoris sequence that also $H_4(\mathcal{X} \setminus \{p_N, p_S\}, \mathbb{Z}) \cong \mathbb{Z}$. On removing the two points, the image of the S^4 zero-section in $\mathcal{X} \setminus \{p_N, p_S\}$ is $I \times S^3$, where I is an interval. Thus the image of this S^4 naturally gives a relative class in $H_4(\mathcal{X} \setminus \{p_N, p_S\}, S^7 \amalg S^7, \mathbb{Z})$, although again it is simple to show that this is isomorphic to $H_4(\mathcal{X} \setminus \{p_N, p_S\}, \mathbb{Z})$ and thus the relative class is represented by a closed 4-cycle also.

Consider a $\mathbb{Z}_k$-invariant closed four-form G on $\mathcal{X}$ that has non-zero integral over the S^4. Then one obtains a four-form on $(\mathcal{X} \setminus \{p_N, p_S\})/\mathbb{Z}_k$ with non-zero integral over $I \times S^3/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts along the Hopf fibre of the S^3. We now normalize the flux $G/(2\pi l_p)^3$ to have period $\tilde{M} \in \mathbb{Z}$ through this (relative) 4-cycle. It follows that lifting to the covering space $\mathcal{X}$, we obtain a period $k\tilde{M}$ through S^4. Then the integral of $(2\pi l_p)^{-6}\frac{1}{2}G \wedge G$ over the covering spacetime $\mathcal{X}$ may be carried out as in the smooth case, to give $\frac{1}{2} \cdot (k\tilde{M}) \cdot \frac{1}{2}(k\tilde{M}) = k^2M^2$. Thus on the quotient $\mathcal{X}/\mathbb{Z}_k$ we obtain

$$N = \frac{1}{(2\pi l_p)^6} \int_{V_{5,2}/\mathbb{Z}_k} *G = \frac{1}{(2\pi l_p)^6} \int_{\mathcal{X}/\mathbb{Z}_k} \frac{1}{2} G \wedge G = kM^2 \,. \tag{5.27}$$

Similarly, we have

$$\frac{1}{(2\pi l_p)^3}\int_{\mathbb{R}^4_{\mathrm{fibre}}/\mathbb{Z}_k} G = \frac{1}{(2\pi l_p)^3}\int_{\Sigma^3} C = M\,, \tag{5.28}$$

where we have noted that the generator Σ^3 of $H_3(V_{5,2}/\mathbb{Z}_k,\mathbb{Z}) \cong \mathbb{Z}_{2k}$ is given by a copy of the boundary of the $\mathbb{R}^4/\mathbb{Z}_k$ fibre of $T^*S^4/\mathbb{Z}_k$ over the north pole $p_N \in S^4$. Comparing to (3.9), we see that $l \cong 0 \bmod 2k$ at infinity, and hence there are *no fractional M5-branes*. Clearly, this is in stark contrast to the Klebanov-Strassler solution.

6 The deformation in the field theory

The deformed supergravity background that we have discussed is of a type which has no known counterpart in the context of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. This was already noticed in [15, 19, 42]. The UV region is asymptotic to a Freund-Rubin background $\mathrm{AdS}_4 \times Y^7$, and thus according to the AdS/CFT dictionary it should be dual to the conformal Chern-Simons-quiver theory extensively discussed in the paper. On the other hand, in the IR region the solution is smooth and displays a finite-sized minimal submanifold at the bottom of the throat. Therefore, according to the general rules of gauge/gravity duality, the dual field theory should have a mass gap and is presumably confining [43]. Understanding the precise mechanism in the field theory is clearly an interesting challenge. In this final section we take a few steps in this direction, leaving a more detailed investigation for future work.

6.1 The field theory in the UV

As we have already explained, at infinity the deformed solution approaches the $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ background. Since $H^4(V_{5,2}/\mathbb{Z}_k,\mathbb{Z}) \cong \mathbb{Z}_{2k}$, at infinity we can only have a flat torsion G-flux of $[G] = l \bmod 2k$. A careful examination of flux quantization in the deformed solution leads to $2M$ units of G-flux through the minimal four-cycle $S^4/\mathbb{Z}_k$ at the zero-section $r = 0$. However, this G-field descreases as we move towards the UV, eventually disappearing at infinity $r = \infty$. The topological class of this G-flux at infinity is $[G] = 0$, while the flux of $*G$ through $V_{5,2}$ is $N = kM^2$. This leads us to conjecture that the field theory in the UV is the superconformal Chern-Simons-quiver theory with gauge group

$$\mathrm{U}(kM^2)_k \times \mathrm{U}(kM^2)_{-k}\ . \tag{6.1}$$

Note that the ranks of the gauge groups could receive subleading corrections that may be important for a consistent interpretation.

On general grounds, the field theoretic interpretation of the deformation is either a perturbation by a relevant operator in the Lagrangian, or involves spontaneous symmetry breaking. These two possibilities are distinguished by the asymptotic behaviour of perturbations in AdS_4. In order to use the AdS/CFT dictionary we need to write the AdS_4 metric in Fefferman-Graham coordinates

$$\mathrm{d}s^2(\mathrm{AdS}_4)_{\mathrm{FG}} = \frac{1}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x_\mu \mathrm{d}x^\mu\right)\,, \tag{6.2}$$

by changing coordinates $\rho^2 = 1/z$. Here recall that ρ is related asymptotically to r via the change of variable (5.6). In particular, for scalar modes we then have

$$\varphi \sim \hat{\varphi} z^{\Delta} + \varphi_0 z^{3-\Delta}\,, \tag{6.3}$$

with φ_0 corresponding to perturbing by an operator of dimension Δ, and $\hat{\varphi}$ corresponding to the VEV of such an operator. Aided by the map between chiral multiplets in the SCFT and modes in the Kaluza-Klein spectrum on $V_{5,2}$, discussed earlier, we will see that the former possibility is realized.

To see this, we examine the leading behaviour of the G-field at infinity, and the corresponding pseudoscalar mode in AdS_4. We may discuss this in the context of general Sasaki-Einstein solutions and then specialize to the case of interest. Consider a self-dual harmonic G-flux in the Calabi-Yau cone background $\mathbb{R}^{1,2} \times C(Y)$, of the form

$$G = \alpha = \mathrm{d}(\rho^{-\nu}\beta)\,, \tag{6.4}$$

where ρ is the radial variable on the cone. This implies $\Delta_Y \beta = \nu^2 \beta$, where Δ_Y is the Laplace operator on Y acting on three-forms. For the associated $\mathrm{AdS}_4 \times Y$ solution, we may then consider a fluctuation of the type $\delta C = \pi \cdot \beta$. It was shown in [44] that this leads to a pseudoscalar field π in AdS_4 with mass[17]

$$m^2 = \frac{\nu(\nu-6)}{4}\,. \tag{6.5}$$

Substituting this into the formula for the dimension of the dual operator, $\Delta(\Delta-3) = m^2$, we obtain $\Delta_\pm = \frac{1}{2}(3 \pm |3-\nu|)$. Which branch to pick depends *a priori* on the specific operator we consider. Going back to our particular $G = \alpha$ given by (5.12), we see that

$$\beta \propto \left(3\tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{\sigma}_3 + \frac{1}{2}\epsilon_{ijk}\sigma_i \wedge \sigma_j \wedge \tilde{\sigma}_k\right)\,, \tag{6.6}$$

and $\nu = 4/3$. Then $\Delta_+ = 3 - \frac{\nu}{2} = \frac{7}{3}$, while $\Delta_- = \frac{\nu}{2} = \frac{2}{3}$. Now, going through all the pseudoscalar modes undergoing shortening conditions in the tables in [11], we find a mode with $\Delta = \frac{7}{3}$ while the other possibility is not realized. In particular, this mode arises as the pseudoscalar component of the chiral operators with dimensions $\Delta = \frac{2}{3}m + 1$, with $m = 2$, that we discussed in section 3.3. From the asymptotic scaling $\alpha \sim z^{2/3}$, we conclude that this operator is in fact added to the Lagrangian (see also [19]).

Since this is the pseudoscalar component of a chiral superfield, we see that it is a Fermionic mass term $\psi^\alpha \psi_\alpha$. This breaks parity invariance, which is reflected in the gravity solution in the presence of the internal flux, the latter being odd under parity. In general, such mass terms may be added to the Lagrangian, in a supersymmetric way, by a quadratic

[17] The reader should not confuse the mass m^2 here with the paramter m in the deformed solution.

superpotential deformation[18]

$$\delta W = \mu \mathrm{Tr}[\phi^2] \qquad \Rightarrow \qquad \delta\mathcal{L} = -\frac{1}{2}\frac{\partial^2 \delta W}{\partial\phi_i\partial\phi_j}\psi_i^\alpha\psi_{j\,\alpha} + \dots \ . \tag{6.7}$$

A priori, we have three such possible mass terms, compatible with the $\mathrm{SU}(2)_r$ global symmetry of the deformed background, namely

$$\delta W = \frac{\mu_+}{2}\left(\mathrm{Tr}[\Phi_1^2] + \mathrm{Tr}[\Phi_2^2]\right) + \frac{\mu_-}{2}\left(\mathrm{Tr}[\Phi_1^2] - \mathrm{Tr}[\Phi_2^2]\right) + \mu_3 \mathrm{Tr}[A_1B_1 + A_2B_2] \ . \tag{6.8}$$

where in the above we mean superfields.

We may deduce which terms are present by analysing more carefully the symmetries of the deformed solution. Recall from section 2.1 that in the undeformed field theory we have a $\mathbb{Z}_2^{\mathrm{flip}}$ symmetry that exchanges $\Phi_1 \leftrightarrow \Phi_2$, $A_i \leftrightarrow B_i$. The generator acts on the z_i coordinates, introduced just below equation (2.8), as $(z_0, z_1, z_2, z_3, z_4) \to (-z_0, z_1, -z_2, z_3, -z_4)$. Hence $\mathbb{Z}_2^{\mathrm{flip}} \subset O(5)$ acts on the deformed quadric (5.1). The internal G-flux then *breaks* this $\mathbb{Z}_2^{\mathrm{flip}}$ symmetry. To see this, notice that for $k = 1$ the zero-section of $\mathcal{X} = T^*S^4$ is S^4, embedded in $\mathbb{R}^5$ by the real parts of the z_i coordinates in (5.1). The volume form on S^4 may be written

$$\mathrm{vol}(S^4) = \frac{1}{4!}\epsilon_{ijklm} z_i \mathrm{d}z_j \wedge \mathrm{d}z_k \wedge \mathrm{d}z_l \wedge \mathrm{d}z_m \,|_{\{\sum_{i=0}^4 z_i^2 = \gamma^2, \ \ z_i\in\mathbb{R}\}} \ . \tag{6.9}$$

This hence changes sign under the generator of $\mathbb{Z}_2^{\mathrm{flip}}$. Now since $\mathbb{Z}_2^{\mathrm{flip}}$ is an isometry, it necessarily maps L^2 harmonic forms to L^2 harmonic forms, and as mentioned earlier the results of [40] imply that $G_{\mathrm{int}} \propto \alpha$ (5.11), where α is given by (5.12), is the only such form. Thus the generator of $\mathbb{Z}_2^{\mathrm{flip}}$ maps $\alpha \mapsto \pm\alpha$. But since α restricts to the volume form on S^4 at $r = 0$, we see that the generator of $\mathbb{Z}_2^{\mathrm{flip}}$ maps $\alpha \mapsto -\alpha$, and thus $G_{\mathrm{int}} \mapsto -G_{\mathrm{int}}$. Hence the related superpotential deformation in (6.8) should also be odd. This requires that $\mu_+ = \mu_3 = 0$, leaving precisely the following supersymmetric mass-term

$$W \to W + \frac{\mu}{2}\left(\mathrm{Tr}[\Phi_1^2] - \mathrm{Tr}[\Phi_2^2]\right) \ . \tag{6.10}$$

We may then regard the full superpotential as depending on the two parameters s and μ. Notice that by setting $s = 0$, the mass term μ is precisely that leading to the ABJM theory in the IR, after integrating out the adjoints.

The deformed F-term equations following from the superpotential deformation (6.10) read

$$B_i\Phi_2 + \Phi_1 B_i = 0\,, \tag{6.11}$$

$$\Phi_2 A_i + A_i\Phi_1 = 0\,, \tag{6.12}$$

$$3s\Phi_1^2 + (B_1A_1 + B_2A_2) + \mu\Phi_1 = 0\,, \tag{6.13}$$

$$3s\Phi_2^2 + (A_1B_1 + A_2B_2) - \mu\Phi_2 = 0\ . \tag{6.14}$$

[18] This deformation then introduces various additional terms in the Lagrangian. For example, we have a quadratic term $\mu^2\mathrm{Tr}[\phi^\dagger\phi]$ in the bosonic F-term potential, with dimension $\Delta = 4/3$, as well as linear terms in μ. Presumably these operators may be detected by analysing appropriate linearized perturbations of the background. However, their structure should be constrained by supersymmetry. See [45] for discussion of a related issue in the context of mass deformations of the ABJM theory.

The simple linear change of variable

$$\Phi_1 = \Psi_1 - \frac{\mu}{6s}\,, \qquad \Phi_2 = \Psi_2 + \frac{\mu}{6s} \tag{6.15}$$

then leads to

$$\begin{aligned} B_i\Psi_2 + \Psi_1 B_i &= 0\,, & (6.16)\\ \Psi_2 A_i + A_i\Psi_1 &= 0\,, & (6.17)\\ 3s\Psi_1^2 + (B_1A_1 + B_2A_2) &= \frac{\mu^2}{12s}\,, & (6.18)\\ 3s\Psi_2^2 + (A_1B_1 + A_2B_2) &= \frac{\mu^2}{12s}\,. & (6.19)\end{aligned}$$

In particular, we see that the Abelian moduli space is exactly the deformed singularity (5.1). The deformation parameter is proportional to the mass, $\gamma^2 = \mu^2/12s$.

6.2 Comments on the field theory in the IR

The supergravity solution implies that the $\mathcal{N} = 2$ superconformal Chern-Simons-matter theory deformed by the mass term will flow in the IR to a *confining* theory. We leave a field-theoretic understanding of this for future work, restricting ourselves here to making only some preliminary comments in this direction.

Firstly, it is instructive to contrast the pattern of $\mathrm{U}(1)_R$ symmetry breaking of our solution with that of the Klebanov-Strassler theory. In the latter case the $\mathrm{U}(1)_R$ symmetry is broken to $\mathbb{Z}_{2M}$ in the UV by the chiral anomaly, and this is then spontaneously broken to $\mathbb{Z}_2$, yielding M vacua. On the gravity side, the breaking of $\mathrm{U}(1)_R$ to $\mathbb{Z}_{2M}$ is reflected by the non-invariance of the fluxes already in the UV [18, 46]. The M vacua are then reflected by the presence of supersymmetric probe branes, representing BPS domain walls interpolating between the vacua. In three dimensions there is no chiral anomaly, and thus $\mathrm{U}(1)_R$ cannot be broken in this way. Indeed, in the supergravity solution we discussed the parameter M is *not* a UV parameter that one can dial at infinity, and in fact the flux vanishes asymptotically. We also expect that no wrapped branes will give rise to BPS domain walls, although we have not checked this.

In analogy with the Klebanov-Strassler cascade, one possible way to interpret the RG flow described by the supergravity solution is to imagine that once the conformal theory is deformed by the mass term in the UV, it starts "cascading", going through a sequence of Seiberg-like dualities where the ranks of the gauge groups decrease, until in the deep IR perhaps one gauge group disappears, and the low energy-theory confines. This idea has recently been suggested in [32, 47] in the context of ABJM-like theories, although the models studied in these references are different from our models. This interpretation is motivated by the brane creation mechanism that we discussed in section 4.4, and by the fact that in the solution there is a varying B_2-field (in the Type IIA reduction). More precisely, the B_2-field suggests that as we proceed to the IR, the NS5-branes rotate around the circle. Taking this point of view, and applying the duality rule of section 4.4, we end up in the IR with a gauge group $\mathrm{U}(-kM)_k \times \mathrm{U}(kM)_{-k}$ after M steps, which clearly doesn't make sense since one gauge group has negative rank. (We could of course stop applying the duality at the previous step.) Notice, however, that what is the precise gauge group in the IR depends on the starting point in the UV, which in turn depends on subleading corrections to kM^2. In any case, it is not clear whether applying this rule is correct, once we turn on the mass deformation. In fact, more conservatively, given the mass term one should integrate out the heavy degrees of freedom, and obtain an effective low-energy theory in the IR. In principle this theory should then exhibit confinement (without supersymmetry breaking). Integrating out the Fermions would *a priori* lead to a possible shift of the Chern-Simons levels. However, because the Fermions are in the adjoint representation in fact the levels are not shifted. Indeed, we have already noted that the mass term is exactly the same mass term which produces the ABJM theory at low energy, starting from the Chern-Simons theory in figure 1 with $s = 0$. Integrating out the bosonic components of the chiral fields in the mass-deformed $n = 2$ theory, the effective superpotential for the low-energy fields A_i, B_i results in a non-local expression, involving square roots of polynomials in these fields. Hopefully, further work along these lines will lead to a precise identification of the IR field theory.

7 Conclusions

In this paper we have constructed a new example of $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality by proposing a simple $\mathcal{N} = 2$ Chern-Simons-matter quiver field theory as the holographic dual to the $\mathrm{AdS}_4 \times V_{5,2}/\mathbb{Z}_k$ Freund-Rubin background in M-theory. This duality presents several novel aspects. For example, the geometry, and hence the field theory, has an $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)_R$ global symmetry (enhanced to $\mathrm{SO}(5) \times \mathrm{U}(1)_R$ for $k = 1$), and hence these models are non-toric. Examples of AdS/CFT dual pairs of non-toric type, where both sides are known explicitly, are quite rare. This model may be thought of as describing the low-energy theory of multiple M2-branes at a quadric hypersurface singularity. In fact, this is the $n = 2$ member of a family of hypersurface singularities ($\mathcal{A}_{n-1}$ four-fold singularities), labelled by a positive integer n, for which we have also presented the corresponding field theories. However, we have explained that only for $n = 2$ and $n = 1$ do these singularities give rise to Freund-Rubin AdS_4 duals, the $n = 1$ model being the ABJM theory. We note that [12] discussed the larger class of ADE four-fold singularities, and it was shown in this reference that in this class the only cases that can admit Ricci-flat Kähler cone metrics are $\mathcal{A}_0 = \mathbb{C}^4$, $\mathcal{A}_1$ and $\mathcal{D}_4$. It would be interesting to construct Chern-Simons-matter theories dual to other hypersurface singularities, and to see whether the $\mathcal{D}_4$ theory admits a Freund-Rubin holographic dual, analogous to that discussed in this paper.

In this paper we have considered the case where the Chern-Simons levels are equal $k_1 = -k_2 = k$. Relaxing this condition, thus allowing for arbitrary levels, corresponds to deforming the Type IIA solutions that we discussed in section 3.2 by turning on a Romans mass [20]. Such solutions will be similar to those discussed in [51, 52] and it would be interesting to find these solutions explicitly.

Another interesting aspect of the model we discussed is that there exists a deformed supergravity solution, that we have argued corresponds to a particular supersymmetric mass deformation of the conformal theory. This deformation is similar to those studied in [45, 48, 49] and other references. We have seen that this mass term is dual to a harmonic $(2, 2)$, primitive (hence self-dual) G-flux on the Calabi-Yau geometry. Quite recently the authors of reference [50] have shown how self-dual background fluxes induce mass terms in the M2-brane worldvolume action, and it would be interesting to see whether this construction generalizes to $\mathcal{N} = 2$ backgrounds of the type we have studied. In the present context the effect of this mass term is rather different from that in the ABJM model studied in [45, 48, 49]: it deforms the classical moduli space in a way that precisely matches the geometry in the supergravity dual. In particular, the solution develops a finite-sized S^4 in the IR, implying that the theory becomes confining. Motivated by brane constructions, we have briefly discussed how this deformation might be interpreted as a "cascade", analogous to the Klebanov-Strassler cascade. However, further work is needed in order to obtain a more conclusive interpretation of the RG flow, and in particular a clearer understanding of the field theory in the deep IR. We expect a similar story to repeat for other deformed solutions with self-dual G-flux, based on different special holonomy manifolds [15, 42].

Finally, in appendix C we describe a Type IIA reduction of the supergravity solutions that is *different* to that considered in the main text, i.e. we reduce on a different choice of M-theory circle. On general grounds, one expects this to lead to a field theory that is mirror to that considered in section 2 (see, for example, [53]). It would be interesting to study this reduction further.

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A Some cohomology computations

In the main text we have made use of a number of different cohomology groups of the various manifolds we have defined, and also the relations between the groups. In this appendix we present the relevant computations.

We begin by defining a manifold that does *not* appear in the main text: we define $\mathcal{X}_n$ by

$$\mathcal{X}_n = \left\{ \prod_{\gamma=1}^{n} (z_0 - a_\gamma) + \sum_{i=1}^{4} z_i^2 = 0 \right\} \subset \mathbb{C}^5 \,. \tag{A.1}$$

Here the a_γ, $\gamma = 1, \ldots, n$, are real, pairwise non-equal constants, which we order as $a_1 < a_2 < \cdots < a_n$. The manifold $\mathcal{X}_2 = \mathcal{X}$ in the main text, which is the deformation of the quadric singularity. The $\mathcal{X}_n$ are smooth non-compact complex manifolds with boundaries $\partial\mathcal{X}_n = Y_n$, where Y_n is defined by (2.9), (3.1). Indeed, the $\mathcal{X}_n$ are *deformations* of the X_n singularities (2.9).

The cohomology of $\mathcal{X}_n$ was discussed in [13], and we briefly review their analysis. For $\gamma = 1, \ldots, n-1$ we may define a four-sphere S^4_γ by requiring that z_0 is real with $a_\gamma < z_0 < a_{\gamma+1}$, and that the z_i, for $i = 1, \ldots, 4$, are all real or all imaginary, depending on the value of γ mod 2. These $n-1$ four-spheres then generate $H_4(\mathcal{X}_n, \mathbb{Z}) \cong \mathbb{Z}^{n-1} \cong H^4(\mathcal{X}_n, Y_n, \mathbb{Z})$, where the last step is Poincaré-Lefschetz duality. This is the only non-trivial homology group of $\mathcal{X}_n$ (of course $H_0(\mathcal{X}_n, \mathbb{Z}) \cong \mathbb{Z}$). Each four-sphere has self-intersection number 2, since its normal bundle may easily be seen to be T^*S^4 which has Euler number 2, and by construction the intersection number of S^4_γ with $S^4_{\gamma+1}$ is 1, with all other intersection numbers vanishing. Poincaré-Lefschetz duality implies that $H^4(\mathcal{X}_n, Y_n, \mathbb{Z})$ and $H^4(\mathcal{X}_n, \mathbb{Z})$ are dual lattices, where recall that $f : H^4(\mathcal{X}_n, Y_n, \mathbb{Z}) \to H^4(\mathcal{X}_n, \mathbb{Z})$ forgets that a class is relative (has compact support). Thus the above discussion shows that $H^4(\mathcal{X}_n, Y_n, \mathbb{Z}) \cong H_4(\mathcal{X}_n, \mathbb{Z})$, equipped with the intersection form, is the root lattice of $\mathcal{A}_{n-1}$, while $H^4(\mathcal{X}_n, \mathbb{Z})$ is the dual weight lattice.

Notice that in the simple case with $n = 2$, where $\mathcal{X}_2 = \mathcal{X} \cong T^*S^4$, the generator of $H^4(\mathcal{X}_2, Y_2, \mathbb{Z}) \cong \mathbb{Z}$ may be taken to be a compactly supported four-form that has integral one over the fibre (the Thom class of the bundle T^*S^4).

We may now compute the cohomology of $Y_n = \partial\mathcal{X}_n$ using the long exact sequence for the pair $(\mathcal{X}_n, Y_n)$. Since the cohomology groups of both $\mathcal{X}_n$ and $(\mathcal{X}_n, Y_n)$ vanish in all degrees other than the top, middle and bottom, it follows that most of the cohomology of Y_n is also trivial. In fact the only non-trivial cohomology group is $H^4(Y_n, \mathbb{Z})$, which arises from the sequence

$$\cdots \longrightarrow H^4(\mathcal{X}_n, Y_n, \mathbb{Z}) \xrightarrow{f} H^4(\mathcal{X}_n, \mathbb{Z}) \longrightarrow H^4(Y_n, \mathbb{Z}) \longrightarrow H^5(\mathcal{X}_n, Y_n, \mathbb{Z}) \cong 0 \ . \quad \text{(A.2)}$$

This implies that $H^4(Y_n, \mathbb{Z}) \cong H^4(\mathcal{X}_n, \mathbb{Z})/f(H^4(\mathcal{X}_n, Y_n, \mathbb{Z})) \cong \mathbb{Z}_n$, where the last isomorphism follows from the above description of the cohomology groups in terms of the root and weight lattices of $\mathcal{A}_{n-1}$. Of course, by Poincaré duality we also have $H_3(Y_n, \mathbb{Z}) \cong \mathbb{Z}_n$.

In the special case that $n = 2$, of main interest in the text, the long exact homology sequence implies that we may take the boundary S^3 of any fibre $S^3 = \partial\mathbb{R}^4$ of T^*S^4 as the generator of $H_3(Y_2, \mathbb{Z}) \cong \mathbb{Z}_2$. Equivalently, viewing Y_2 as an S^3 bundle over S^4, a copy of the fibre at any point on the base generates this third homology group.

Next we introduce the free circle action on Y_n by $\mathrm{U}(1)_b \cong \mathrm{SO}(2)_{\mathrm{diag}} \subset \mathrm{SO}(4)$, where $\mathrm{SO}(4)$ acts on the coordinates z_i, $i = 1, \ldots, 4$, in the vector representation. The quotient $M_n = Y_n/\mathrm{U}(1)_b$ is then a smooth compact six-manifold. The cohomology of this space may be deduced from the Gysin sequence for the circle fibration of Y_n over M_n:

$$\begin{aligned} \cdots \longrightarrow \quad & H^{i-2}(M_n, \mathbb{Z}) \xrightarrow{\cup c_1} H^i(M_n, \mathbb{Z}) \longrightarrow H^i(Y_n, \mathbb{Z}) \longrightarrow \\ & H^{i-1}(M_n, \mathbb{Z}) \longrightarrow \cdots \ . \end{aligned} \quad \text{(A.3)}$$

It is straightforward to derive this sequence from the long exact sequence for the total space $\mathcal{L}$ of the complex line bundle over M_n associated to the $\mathrm{U}(1)_b$ circle bundle: note that $\mathcal{L}$ has boundary Y_n, and base M_n. One needs to combine this sequence with the Thom isomorphism — this is precisely where the cup product with $c_1 = c_1(\mathcal{L})$ comes from above, since for a complex line bundle c_1 is equal to the Euler class of the underlying rank 2 real vector bundle. The last map in the Gysin sequence (A.3) is just pull-back from M_n to Y_n.

Using the sequence (A.3), together with the known cohomology of Y_n computed above, we may compute the cohomology (and properties of the cohomology ring) of M_n. Since $H^1(Y_n, \mathbb{Z}) \cong H^2(Y_n, \mathbb{Z}) \cong 0$, it follows immediately from $i = 2$ in (A.3) that $c_1 \equiv \Omega_2$ is the generator of $H^2(M_n, \mathbb{Z}) \cong \mathbb{Z}$. Here the notation Ω_2 was introduced in the main text just before equation (3.16). Similarly, $H^3(Y_n, \mathbb{Z}) \cong 0$ implies that $H^3(M_n, \mathbb{Z}) \cong 0$. Then $i = 4$ above implies $\mathbb{Z}_n \cong H^4(Y_n, \mathbb{Z}) \cong H^4(M_n, \mathbb{Z})/[H^2(M_n, \mathbb{Z}) \cup c_1]$. Now, $H^4(M_n, \mathbb{Z}) \cong H_2(M_n, \mathbb{Z})$, so the free part of $H^4(M_n, \mathbb{Z})$ is $\mathbb{Z} \cong H^2(M_n, \mathbb{Z})$ by the Universal Coefficient Theorem. Moreover, the torsion in $H^4(M_n, \mathbb{Z})$ is the torsion in $H_3(M_n, \mathbb{Z})$, but this is Poincaré dual to $H^3(M_n, \mathbb{Z}) \cong 0$. Thus $H^4(M_n, \mathbb{Z}) \cong \mathbb{Z}$, and the Gysin sequence thus tells us that the square of the generator of $H^2(M_n, \mathbb{Z})$ is n *times* the generator of $H^4(M_n, \mathbb{Z})$. We may equivalently state this as

$$\int_{\Sigma^4} \Omega_2 \cup \Omega_2 = n\,, \tag{A.4}$$

where Σ^4 denotes the generator of $H_4(M_n, \mathbb{Z})$, again as in the main text. The result (A.4) follows from Poincaré duality, and the last map in the Gysin sequence that says cupping $H^4(M_n, \mathbb{Z})$ with $c_1 = \Omega_2$ (which is Poincaré dual to Σ^4) maps the generator of $H^4(M_n, \mathbb{Z})$ to the generator of $H^6(M_n, \mathbb{Z}) \cong \mathbb{Z}$. Notice that M_n then has the same cohomology groups as $\mathbb{CP}^3$ (where $M_1 \cong \mathbb{CP}^3$), but that the cohomology *ring* depends on n via the above calculation.

We may now compute the cohomology of the quotient $Y_n/\mathbb{Z}_k$. This is also a smooth seven-manifold, where we take $\mathbb{Z}_k \subset \mathrm{U}(1)_b$. This immediately gives $\pi_1(Y_n/\mathbb{Z}_k) \cong H_1(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_k$. The Gysin sequence (A.3), with $Y_n/\mathbb{Z}_k$ in place of Y_n, now has $c_1 = k\Omega_2$. Precisely as we argued above, this implies the important result that $H^4(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong H^4(M_n, \mathbb{Z})/[H^2(M_n, \mathbb{Z}) \cup k\Omega_2] \cong \mathbb{Z}_{nk}$. Of course, by Poincaré duality also $H_3(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$. Indeed, the Poincaré dual sequence implies that the generator Σ^2 of $H_2(M_n, \mathbb{Z}) \cong \mathbb{Z}$ lifts to the generator Σ^3 of $H_3(Y_n/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{nk}$, where Σ^3 is the total space of the circle bundle over a representative of Σ^2. This was used at the end of section 3.5.

Finally, recall that in the special case of $n = 2$ the generator of $H_3(Y_2, \mathbb{Z}) \cong \mathbb{Z}_2$ can be taken to be a copy of the fibre S^3 in the fibration $S^3 \hookrightarrow Y_2 \to S^4$. The fibres over the poles p_N, p_S of the S^4 are mapped into themselves under $\mathbb{Z}_k$, with the Hopf action of $\mathbb{Z}_k$ on S^3 giving the quotient $S^3/\mathbb{Z}_k$. It then follows from the last paragraph that this Lens space $S^3/\mathbb{Z}_k \cong \Sigma^3$ generates $H_3(Y_2/\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_{2k}$.

B The Stenzel metric

In this appendix we review the construction of the Stenzel metric on $\mathcal{X} \cong T^*S^4$. The deformed quadric $\mathcal{X}$ is defined as

$$\sum_{i=0}^{4} z_i^2 = \gamma^2\,, \tag{B.1}$$

and the Stenzel metric on this may be written by introducing left-invariant one-forms L_{AB} on SO(5), $A, B = 1, \ldots, 5$, satisfying $\mathrm{d}L_{AB} = L_{AC} \wedge L_{CB}$. We split $A = (1, 2, i)$, with $i = 1, 2, 3$, where the L_{ij} are left-invariant one-forms for SO(3), and define

$$\sigma_i = L_{1i}, \qquad \tilde{\sigma}_i = L_{2i}, \qquad \nu = L_{12} \ . \tag{B.2}$$

These are one-forms on the coset space $V_{5,2} = \mathrm{SO}(5)/\mathrm{SO}(3)$. The metric on (B.1) is then [15]

$$\mathrm{d}s^2 = c^2 \mathrm{d}r^2 + c^2 \nu^2 + a^2 \sigma_i^2 + b^2 \tilde{\sigma}_i^2 \ . \tag{B.3}$$

It is useful to introduce the orthonormal frame

$$e^0 = c\mathrm{d}r\,, \qquad e^{\tilde{0}} = c\nu\,, \qquad e^i = a\sigma_i\,, \qquad e^{\tilde{i}} = b\tilde{\sigma}_i \ . \tag{B.4}$$

A holomorphic frame is provided by

$$\epsilon^0 = -e^0 + \mathrm{i}e^{\tilde{0}}\,, \qquad \epsilon^i = e^i + \mathrm{i}e^{\tilde{i}} \ . \tag{B.5}$$

In this frame, we take the Kähler form J and holomorphic $(4,0)$-form Ω to be the standard forms

$$J = \frac{\mathrm{i}}{2} \epsilon^\alpha \wedge \bar{\epsilon}^{\bar{\alpha}}, \qquad \Omega = \epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \ . \tag{B.6}$$

Thus these automatically satisfy the SU(4)-structure algebraic relations $J \wedge \Omega = 0$, $\frac{1}{4!} J^4 = \frac{1}{16} \Omega \wedge \bar{\Omega} = -e^{0\tilde{0}1\tilde{1}2\tilde{2}3\tilde{3}}$. A Ricci-flat Kähler metric requires $\mathrm{d}J = 0 = \mathrm{d}\Omega$. It is straightforward to check that $\mathrm{d}J = 0$ is equivalent to the ordinary differential equation (ODE)

$$(ab)' = c^2\,, \tag{B.7}$$

where a prime denotes differentiation with respect to r, while imposing $\mathrm{d}\Omega = 0$ is equivalent to the four ODEs

$$\begin{aligned} 3\frac{a'}{a} + \frac{c'}{c} - 3\frac{b}{a} &= 0\,, \\ 3\frac{b'}{b} + \frac{c'}{c} - 3\frac{a}{b} &= 0\,, \\ 2\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} - 2\frac{b}{a} - \frac{a}{b} &= 0\,, \\ 2\frac{b'}{b} + \frac{a'}{a} + \frac{c'}{c} - 2\frac{a}{b} - \frac{b}{a} &= 0 \ . \end{aligned} \tag{B.8}$$

Although this naively looks overdetermined, it is simple to check by taking linear combinations that these five ODEs are equivalent to the three ODEs

$$\begin{aligned} \frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{2ab}\,, \\ \frac{b'}{b} &= \frac{a^2 + c^2 - b^2}{2ab}\,, \\ \frac{c'}{c} &= \frac{3(a^2 + b^2 - c^2)}{2ab} \ . \end{aligned} \tag{B.9}$$

This is the same system of equations that were presented in [15], although in the latter reference they were derived by first finding the second order Einstein equations, and then constructing a superpotential. Here we have derived them directly from the Ricci-flat Kähler conditions. A solution to these equations, which is a smooth complete metric on $\mathcal{X} = T^*S^4$, was found by Stenzel [54]. This is the solution written in (5.4).

C A different reduction to Type IIA

In sections 3.2 and 3.5 we considered reducing M-theory on $\mathbb{R}^{1,2}\times X_n/\mathbb{Z}_k$ with N spacefilling M2-branes, or its near-horizon limit $\mathrm{AdS}_4 \times Y_n/\mathbb{Z}_k$, along $\mathrm{U}(1)_b$ to Type IIA string theory. Recall here that X_n admits a Ricci-flat Kähler cone metric only for $n = 1$ and $n = 2$. In the case $n = 2$, one problem with this Type IIA reduction is that as soon as one deforms the $\mathrm{AdS}_4 \times Y_2/\mathbb{Z}_k$ solution to the $\mathbb{R}^{1,2} \times \mathcal{X}_2/\mathbb{Z}_k$ solution, the reduction along $\mathrm{U}(1)_b$ is no longer well-behaved. Specifically, the $\mathrm{U}(1)_b$ action fixes the north and south poles of the S^4 zero-section of $\mathcal{X} \equiv \mathcal{X}_2 \cong T^*S^4$; since these are codimension eight, there is no simple interpretation of the resulting singularity in the dilaton in Type IIA string theory. Thus the Type IIA supergravity solution cannot be trusted in the IR region near to the S^4 at $r = 0$. However, there is a *different* reduction to Type IIA that *is* well-behaved. We briefly describe this here, leaving a more thorough investigation for future work.

Recall that in section 4.2 we introduced a different $\mathrm{U}(1) \equiv \mathrm{U}(1)_6$ action on X_n. If we regard X_n as being defined by the hypersurface equation (2.8), the coordinates $(A_1, A_2, B_1, B_2, z_0 = [s(n+1)]^{1/n}\Phi_2)$ have charges $(1, 0, -1, 0, 0)$ under $\mathrm{U}(1)_6$. In fact, we may *deform* X_n to $\mathcal{X}_n$ given by (A.1), so that $\mathrm{U}(1)_6$ also acts on the smooth manifold $\mathcal{X}_n$. Of course, to obtain a solution to eleven-dimensional supergravity, we should equip $\mathcal{X}_n$ with a Calabi-Yau metric. For $n = 1$, $n = 2$, we may use complete asymptotically conical Calabi-Yau metrics (the flat metric on $\mathcal{X}_1 \cong \mathbb{C}^4$; the Stenzel metric on $\mathcal{X}_2 \cong T^*S^4$). These are the metrics relevant for application to the AdS/CFT correspondence. Such metrics do not exist for $n > 2$, in which case the reader can imagine that (A.1) is a local model in a *compact* Calabi-Yau manifold. Yau's theorem will then give a Ricci-flat Kähler metric on this space which is incomplete at the boundary. In any case, the precise details of the metric will not be important in what follows.

Consider reduction of M-theory on $\mathbb{R}^{1,2} \times \mathcal{X}_n$, with N spacefilling M2-branes, along $\mathrm{U}(1)_6$. The fixed point set is codimension four, namely $\{A_1 = B_1 = 0\}$, which cuts out the locus

$$\prod_{\gamma=1}^{n}(z_0 - a_\gamma) + A_2 B_2 = 0 \; . \tag{C.1}$$

This is the deformation of the $\mathcal{A}_{n-1}$ singularity: it has $n-1$ two-spheres S^2_γ, defined similarly to the four-spheres S^4_γ in appendix A, that intersect according to the root lattice of $\mathcal{A}_{n-1} = \mathrm{SU}(n)$. This becomes a D6-brane locus when we reduce to Type IIA. Indeed, the Type IIA spacetime is flat, since $\mathcal{X}_n/\mathrm{U}(1)_6 \cong \mathbb{R}^7$. To see this, note that $\mathcal{X}_n/\mathbb{C}^*_6$ is described by

$$z + \prod_{\gamma=1}^{n} (z_0 - a_\gamma) + A_2 B_2 = 0 \; . \tag{C.2}$$

where $z = A_1 B_1$. This is simply $\mathbb{C}^3$. The quotient space is thus diffeomorphic to $\mathbb{R}^7 \cong \mathbb{R}_7 \times \mathbb{C}^3$, where $\mathbb{R}_7$ is spanned by $|A_1|^2 - |B_1|^2$, which one can think of as the moment map for $\mathrm{U}(1)_6$, and $\mathbb{C}^3$ is spanned by (A_2, B_2, z_0). The fixed point locus is thus at the origin of $\mathbb{R}_7$, and cuts out the hypersurface (C.1) in the $\mathbb{C}^3$ part.

The reduction of $\mathbb{R}^{1,2} \times \mathcal{X}_n$ along $\mathrm{U}(1)_6$ is thus the flat spacetime $\mathbb{R}^{1,9} = \mathbb{R}^{1,2} \times \mathbb{R}_7 \times \mathbb{C}^3$, with N spacefilling D2-branes and a single spacefilling D6-brane sitting at the origin of $\mathbb{R}_7$ and wrapping the divisor (C.1) in $\mathbb{C}^3$. Notice that this description gives the correct amount of supersymmetry, since a D-brane wrapped on a divisor in a three-fold preserves four supercharges, or $\mathcal{N} = 2$ supersymmetry in $d = 3$.

There are $n-1$ four-cycles in $\mathcal{X}_n$, and the quantized G-flux through the generators S^4_γ defined in appendix A gives

$$\frac{1}{(2\pi l_p)^3} \int_{S^4_\gamma} G = M_\gamma \in \mathbb{Z} \; . \tag{C.3}$$

In the Type IIA reduction considered in this section, this is dual to adding M_γ units of *worldvolume* gauge field flux on the D6-brane through the two-sphere S^2_γ in the deformed $\mathcal{A}_{n-1}$ singularity (C.1). A general discussion of this may be found in [55]. Thus

$$\frac{1}{2\pi l_s g_s} \int_{S^2_\gamma} F = M_\gamma \, , \tag{C.4}$$

where F is the U(1) gauge field on the D6-brane.

In the limit where $a_\gamma \to 0$, which is the hypersurface singularity X_n, the D6-brane is wrapped on $\mathbb{R}^{1,2} \times \mathcal{A}_{n-1}$ (we emphasize that the *spacetime* is flat Minkowski spacetime). In particular, for $n = 2$ we have an $\mathcal{A}_1$ singularity, although for $n > 2$ the above analysis shows that the $\mathcal{A}_1$ *quiver* in section 2 is *not* related to this $\mathcal{A}_1$ singularity in the Type IIA reduction on $\mathrm{U}(1)_6$. Indeed, since we are reducing on a different circle, one expects the effective gauge theory derived from the brane configuration described above to be *mirror* to the gauge theory in section 2, which we derived from the Type IIA reduction on $\mathrm{U}(1)_b$ in section 3.5.

We may also consider taking the $\mathbb{Z}_k$ quotient along $\mathrm{U}(1)_b$. The charges of the coordinates $(A_1, A_2, B_1, B_2, z_0)$ under $\mathrm{U}(1)_b$ are $(1, 1, -1, -1, 0)$, and thus in the Type IIA internal space $\mathbb{R}_7 \times \mathbb{C}^3$, spanned by the moment map $|A_1|^2 - |B_1|^2$ and (A_2, B_2, z_0), respectively, $\mathrm{U}(1)_b$ acts with charges $(1, -1, 0)$ on $\mathbb{C}^3$. Thus the $\mathbb{Z}_k$ quotient along $\mathrm{U}(1)_b$ leads to a $\mathbb{Z}_k$ *singularity in spacetime*, or more precisely an $\mathcal{A}_{k-1}$ singularity. This would usually lead to an $\mathrm{SU}(k)$ gauge symmetry in the transverse six-dimensional space. Contrast this with the $\mathcal{A}_{n-1}$ singularity on which the D6-brane is wrapped.

Finally, notice that we may perform a T-duality along the U(1) which acts with charges $(1, -1)$ on the coordinates (A_2, B_2). This gives a Type IIB brane set-up where the spacetime is $\mathbb{R}^{1,2}\times\mathbb{R}_7\times S^1\times\mathbb{R}^5$, with N spacefilling D3-branes wrapping the S^1 circle (that arises from the T-duality). Here $\mathbb{R}^5$ arises as $\mathbb{R}^5=\mathbb{R}\times\mathbb{C}^2$, where $\mathbb{R}$ is spanned by the moment map $|A_2|^2-|B_2|^2$, and $\mathbb{C}^2$ is spanned by (z_0, A_2B_2). Since the fixed point locus is $\{A_2=B_2=0\}$, which is a copy of $\mathbb{R}^{1,2}\times\mathbb{R}_7\times\mathbb{C}$ in the IIA spacetime (with $\mathbb{C}$ spanned by the coordinate z_0), on T-dualizing this becomes a linearly embedded spacefilling NS5-brane. More precisely, the NS5-brane wraps the $\mathbb{R}_7$ direction, sits at a point in S^1, and wraps the copy of $\mathbb{C}\subset\mathbb{R}^5$ spanned by the coordinate z_0. When we divide by $\mathbb{Z}_k\subset \mathrm{U}(1)_b$, the fixed locus is precisely the $\mathcal{A}_{k-1}$ singularity, and we thus obtain k linearly embedded spacefilling NS5-branes in the Type IIB dual. The spacefilling D6-brane wrapped on the deformation of the $\mathcal{A}_{n-1}$ singularity becomes a spacefilling D5-brane wrapped on a non-linearly embedded copy of $\mathbb{R}^3$ in $\mathbb{R}^5$. This is because the four-manifold (C.1) fibres over $\mathbb{R}^3$ with n fixed points. The two copies of $\mathbb{R}^3$ wrapped by the D5-brane and the k NS5-branes thus intersect at n points in $\mathbb{R}^6=\mathbb{R}_7\times\mathbb{R}\times\mathbb{C}^2$.

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JHEP12(2009)017

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